1. Matrix

Definition of a Matrix :- A matrix is a rectangular arrangement of rows & columns which is represented as :-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Element of a Matrix :- The numbers a_{11}, a_{12}, a_{13} ---- etc. are known as the elements of the matrix. which is represented as a_{ij}. which denotes element in i^{th} row & z column.

Order of a matrix :- The no. of rows & columns of a matrix determines the order of the matrix.

Hence, a matrix having m rows and n columns is said to be of the order $m \times n$.

Types of Matrices :- i) **Row matrix :-** A matrix having only one row and any number of columns.

Eg:- $A = [A \ B \ C \ D]$

$A = [1 \ 2 \ 3 \ 4]$

Column Matrix :- A matrix having only one column and any number of rows is called column Matrix.

Eg:- $A = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$ $A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Square Matrix :- A matrix in which the number of rows is equal to the no. of columns is called a square matrix.

Eg :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Rectangular Matrix :- A matrix in which the number of rows is not equal to the no. of columns is called a rectangular matrix.

Eg :- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}_{2\times 4}$

Null Matrix :- A matrix of any order having all the elements are zero is called null matrix or zero matrix.

Eg :- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

Identity Matrix (or) Unit Matrix :- A square matrix whose each diagonal element is unity and all other elements are zero is called Identity (or) Unit Matrix.

Eg :- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Diagonal Matrix :- A square matrix in which diagonal element are non-zero and all non diagonal elements are zero's is called diagonal matrix.

Eg :- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$

Scalar Matrix :- A square matrix in which any ~~non~~ every non diagonal element is zero and all diagonal matrix are equal is called scalar matrix.

Eg :- $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$

Uppertriangular Matrix :- The Uppertriangular matrix has all the elements below the main diagonal as zero.

Eg :- $\begin{bmatrix} 1 & 2 & 3 \\ 9 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3\times 3}$

Lower triangular matrix :- The lower triangular matrix has all the elements above the main diagonal as zero.

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}_{3\times3}$

Singular Matrix :- A square matrix A is said to be singular matrix if determinent of matrix A is zero i.e $|A| = 0$. Otherwise it is non singular.

Symmetrix Matrix :- If a matrix is equal to its transpose then it is called symmetrix Matrix i.e $\boxed{A^T = A}$

Skew Symmetrix Matrix :- If the transpose of a given matrix is equal to its additive inverse then those matrix is called skew symmetrix Matrix i.e $\boxed{A^T = -A}$

Orthogonal Matrix :- A square matrix of order "n" is said to be orthogonal if $\boxed{AA^T = A^TA = I}$

Idempotent Matrix :- A square matrix "A" is said to be Idempotent if $\boxed{A^2 = A}$

Involutory matrix :- A square matrix "A" is said to be involutory matrix if $\boxed{A^2 = I}$

Equal Matrix :- If A, B are any two matrices then the order of both matrices are same and corresponding elements of the matrices are equal.

Eg:- $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$

$x = 1, y = 2, z = 3, w = 4$

Rank of a Matrix :- The rank of a matrix is the order of the highest order non zero minor.

Let us consider a non zero matrix "A".

A real number "r" is said to be the rank of matrix "A". If it satisfies the following condition.

1) Every minor of order $r+1$ is zero.
2) Their exists atleast one minor of order "r" that is non zero.

③ The rank of a "A" matrix is denoted by rho of "A".

Rank of a Matrix By Echelone form :- The rank of matrix in Echelone form in is the number of non zero rhos.

1) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ into Echelone form and find its rank.

sol

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & -3 & -5 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{bmatrix}$$

$R_3 \to 2R_3 + 3R_2$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

i.e, $\rho(A) = 3$

No. of non zero rows is 3

∴ Rank of $A = 3$

2) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ into Echelone form find its rank.

sol

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of non zero rows is 2

Rank of $A = 2$

i.e, $\boxed{\rho(A) = 2}$

3) Reduce the matrix $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ into Echelone form find its rank.

<u>sol</u> Given

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_4 \to 2R_4 + R_3$

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to 3R_3 + 2R_2$

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ No. of non-zeroes is 2

$\therefore$ Rank of non - zeroes is 2

Rank of a matrix is 2

$\boxed{\rho(A) = 2}$

4) Reduce the rank matrix $A=\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ into Echelone form find its rank.

Sol Given

$$A=\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$A=\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$\therefore$ Rank of $A = 3$

i.e, $\boxed{\rho(A) = 3}$

5) Reduce the rank matrix $A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into Echelone form. find its rank

Sol Given

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$R_4 \to R_4 - 6R_1$

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of non zero rows is 3

Rank of $A = 3$

i.e, $\boxed{\rho(A) = 3}$

5) Reduce the matrix $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ into Echelone form. find its Rank.

sol Given

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 0 & 0 & -10 & -14 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & -10 & -14 \end{bmatrix}$$

$R_4 \to R_4 + 2R_3$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ No. of non-zeroes is 2

∴ Rank of non-zeroes is 2

Rank of a matrix is 2

$$\boxed{\rho(A) = 2}$$

7] Reduce the matrix $A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ into Echelone form find its Rank.

sol Given

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 0 & 1 & 5 & 9 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 5 & 9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$\therefore$ Rank of $A = 3$

i.e, $\boxed{\rho(A) = 3}$

8) Reduce the matrix $A = \begin{bmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$ into Echelone form find its Rank.

sol Given

$$A = \begin{bmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

$R_2 \to 4R_2 + 6R_1$

$$A = \begin{bmatrix} 4 & -6 & 0 \\ 0 & -36 & 4 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

$R_3 \to 4R_3 + R_2$

$$A = \begin{bmatrix} 4 & -6 & 0 \\ 0 & -36 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

$R_4 \leftrightarrow R_3$

$$A = \begin{bmatrix} 4 & -6 & 0 \\ 0 & -36 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to 36R_3 + R_2$

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & -36 & 4 \\ 0 & 0 & 148 \\ 0 & 0 & 0 \end{bmatrix}$$

No. of non zero rows is 3

Rank of $A = 3$

$\rho(A) = 3$

9) Reduce the matrix $A=\begin{bmatrix}0&16&8&4\\2&4&8&16\\16&8&4&2\\4&8&16&2\end{bmatrix}$ into Echelone form and find its Rank.

sol Given

$$A=\begin{bmatrix}0&16&8&4\\2&4&8&16\\16&8&4&2\\4&8&16&2\end{bmatrix}.$$

$R_4 \to R_4-2R_2$

$$A=\begin{bmatrix}0&16&8&4\\2&4&8&16\\16&8&4&2\\0&0&0&-30\end{bmatrix}$$

$R_3 \to 2R_3-R_1$

$$A=\begin{bmatrix}0&16&8&4\\2&4&8&16\\32&0&0&0\\0&0&0&-30\end{bmatrix}$$

$\therefore$ Rank of $A=4$

i.e. $\boxed{\rho(A)=4}$

10) Reduce the matrix $A=\begin{bmatrix}1&2&3\\1&4&2\\2&6&5\end{bmatrix}$ into Echelone form. find its Rank.

sol Given

$$A=\begin{bmatrix}1&2&3\\1&4&2\\2&6&5\end{bmatrix}$$

$R_3 \to R_3-2R_1$

$$A=\begin{bmatrix}1&2&3\\1&4&2\\0&2&-1\end{bmatrix}$$

$R_2 \to R_2-R_1$

$$A=\begin{bmatrix}1&2&3\\0&2&-1\\0&2&-1\end{bmatrix}$$

$R_3 \to R_3-R_2$

$$A=\begin{bmatrix}1&2&3\\0&2&-1\\0&0&0\end{bmatrix}$$

No. of non zero rows is 2

Rank of A is 2

$\therefore \boxed{\rho(A)=2}$

11) Reduce the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 6\\ 3 & 9 & 12\end{bmatrix}$ into Echelone form. find its Rank.

sol Given $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 6\\ 3 & 9 & 12\end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 6\\ 0 & 3 & 3\end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 3 & 3\end{bmatrix}$

$R_3 \to R_3 - 3R_2$

$=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}$

$\therefore$ Rank of $A=3$

i.e $\boxed{\rho(A)=3}$

12) Reduce the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 7\\ 3 & 6 & 10\end{bmatrix}$ into Echelone form. find its Rank.

sol Given $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 7\\ 3 & 6 & 10\end{bmatrix}$

$R_3 \to R_3 - 3R_1$

$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 7\\ 0 & 0 & 1\end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}$

$R_3 \to R_3 - R_2$

$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$

$\therefore$ No. of non zero rows is 2

Rank of A is 2

$\therefore \boxed{\rho(A)=2}$

13) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ into Echelone form. find its rank.

Sol Given

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$R_3 \to 4R_3 + R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

$\therefore$ Rank of $A = 3$

i.e $\boxed{\rho(A) = 3}$

14) Reduce the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ into Echelone form. find its rank.

Sol Given

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -6 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ No. of non zero rows is 2

Rank of $A = 2$

i.e $\boxed{\rho(A) = 2}$

15. Reduce the matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ into Echelone form find its rank.

sol Given

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 0 & 2 & 4 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ No. of non zero rows is 2

Rank of $A = 2$

i.e $\boxed{\rho(A) = 2}$

Finding Rank by using Normal form

Normal form :- Every $m \times n$ of rank r can be reduced to the form

1) $I_r - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (or) ② $\begin{bmatrix} I_r & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ③ $\begin{bmatrix} I_r \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (or)

④ $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

By a finite ~~chang~~ chain of elementary row (or) column operations where I_r is the r-rowed unit matrix.

The above form is called normal form (or) first canonical form of a matrix.

~~Problems:- Reduce the matrix A=~~

① Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form. And find its Rank.

sol Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$R_2 \to R_2 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 4C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$R_3 \to R_3 \times (-1/12)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 3C_2 - 2C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 \times (-1/3)$

$C_4 \to C_4 \times (-1/5)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1/5 \end{bmatrix}$$

$C_4 \to C_4 - C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 \times (-5)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \leftrightarrow C_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \ldots$$

No. of non zeroes row is 3

$\boxed{\rho(A) = 3}$

(2) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

sol Given

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$C_3 \to C_3 - 2C_1$

$C_2 \to C_2 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

No. of non zeroes rows is 2

$\boxed{\rho(A) = 2}$

(3) $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -3 \end{bmatrix}$

sol $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -3 \end{bmatrix}$

$R_3 \to 2R_3 - R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 0 \\ 0 & -4 & 1 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 1 & -6 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 1 & -6 \end{bmatrix}$$

$C_2 \to C_2 \times (-1/4)$

$C_4 \to C_4 \times (-1/6)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \end{bmatrix}$$

No. of non zeroes rows is 3

$\boxed{\rho(A) = 3}$

(4) find $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

sol

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 0 & 2 & 4 & -4 \end{bmatrix}$$

$C_3 \to C_3 + C_4$

$$A = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 4 & 0 & 8 & 6 \\ 0 & 2 & 0 & -4 \end{bmatrix}$$

$C_3 \to C_3 - 2C_1$

$$A = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 4 & 0 & 0 & 6 \\ 0 & 2 & 0 & -4 \end{bmatrix}$$

$C_4 \to C_4 + 2C_2$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$C_4 \to 2C_4 - 3C_1$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$C_1 \to C_1 \times (1/4)$

$R_4 \to R_4 \times (1/2)$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2r \\ 0 \end{bmatrix}$$

∴ No. of non zero row is 2

$\boxed{\rho(A) = 2}$

(5) $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$

sol Given

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$R_2 \to R_2 - 2R_1$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & -3 \\ 0 & -2 & -7 \end{bmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_2 + 2C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -3 \\ 0 & -2 & -7 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & -2 & -5 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -15 \end{bmatrix}$$

$R_2 \to R_2/-3$, $R_3 \; R_3/-15$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ No. of non zero rows is 3.

$\rho(A) = 3$

(6) $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Sol Given

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_1$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 2R_3$

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_1 \times 1/2$

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \times C_3 - 3C_1$

$C_4 \rightarrow C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \times 1/4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \times (-1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ No. of non zero rows is 3

$$\boxed{\rho(A) = 3}$$

⑦ find $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

Sol Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$$

$R_4 \to R_4 + 7R_2$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 6 & -30 \end{bmatrix}$$

$R_4 \to R_4 / 6$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$C_2 \to C_2 - 1$

$C_3 \to C_3 - C_1$

$C_4 \to C_4 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$C_4 \to C_4 + 5C_2$

$C_3 \to C_3 - C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & +0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$\cancel{C_4 \to C_4 + 2C_2}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

$C_4 \to C_4 + 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$C_4 \to C_4 \times (-1/3)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No. of non zero rows is 4

$\boxed{\rho(A) = 4}$

8) find P $A=\begin{bmatrix}2&3&-1&1\\1&-1&-2&-4\\3&1&3&-2\\6&3&0&-1\end{bmatrix}$

sol Given

$$A\begin{bmatrix}2&3&-1&1\\1&-1&-2&-4\\3&1&3&-2\\6&3&0&-1\end{bmatrix}$$

$R_2\to 2R_2-R_1$

$R_3\to 2R_3-3R_1$

$R_4\to R_4-3R_1$

$$\begin{bmatrix}2&3&-1&1\\0&-5&-3&-9\\0&-7&9&-7\\0&-6&3&-4\end{bmatrix}$$

$C_4\to C_4+C_3$

$$\begin{bmatrix}2&3&1&0\\0&-5&-3&-12\\0&-7&9&2\\0&-6&3&-1\end{bmatrix}$$

$C_3\to 2C_3-C_1$

$$\begin{bmatrix}2&3&0&0\\0&-5&-6&-12\\0&-7&18&2\\0&-6&6&-1\end{bmatrix}$$

$R_4\to R_4+R_2$

$$\begin{bmatrix}2&3&0&0\\0&-5&-6&-12\\0&-7&18&2\\0&-11&0&-13\end{bmatrix}$$

$C_4\to C_4-2C_3$

$$\begin{bmatrix}2&3&0&0\\0&-5&-6&0\\0&-7&18&-34\\0&-11&0&-13\end{bmatrix}$$

$R_3\to R_3+3R_2$

$$\begin{bmatrix}2&3&0&0\\0&-5&-6&0\\0&-26&0&-34\\0&-11&0&-13\end{bmatrix}$$

$C_2\to 2C_2-3C_1$

$$\begin{bmatrix}2&0&0&0\\0&-10&-6&0\\0&-52&0&-34\\0&-22&0&-13\end{bmatrix}$$

$C_2\to C_2/-2$

$$\begin{bmatrix}1&0&0&0\\0&5&-6&0\\0&26&0&-34\\0&11&0&-13\end{bmatrix}$$

$C_3\to C_3/-6$

$$\begin{bmatrix}1&0&0&0\\0&5&1&0\\0&26&0&-34\\0&11&0&-13\end{bmatrix}$$

$C_2\to C_2-5C_3$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&26&0&-34\\0&11&0&-13\end{bmatrix}$$

$C_2\to 13C_2+11C_4$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&338&0&-34\\0&0&0&-13\end{bmatrix}$$

$R_3\to R_3/2$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&129&0&-17\\0&0&0&-13\end{bmatrix}$$

$C_2\to C_2/129$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&-17\\0&0&0&-13\end{bmatrix}$$

$C_4\to C_4+17C_2$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-13\end{bmatrix}$$

$C_4\to C_4/-13$

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$R_2\leftrightarrow R_3$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$\therefore\ \rho(A)=3$

⑨ find $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

Sol Given

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$C_1 \to C_1 - C_4$

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$C_4 \to C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$C_3 \to C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 6C_3 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 6 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to 10R_3 - 16R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to 6C_4 + 10C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2/6,\ C_3/60.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ No of non zero rows is 3

$\boxed{\rho(A) = 3}$

Couchy Binet formulae

Let AB be the product of A and B matrices then $\det(AB) = \det A \cdot \det B$

Problems

1. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 & 0 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ then S.T $|AB| = |A| \cdot |B|$

sol Given matrices

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 6 & 0 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+9+1 & 12+6+2 & 0+3+3 \\ 1+12+2 & 6+8+4 & 0+4+6 \\ 0+3+1 & 0+2+2 & 0+1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 20 & 6 \\ 15 & 18 & 10 \\ 4 & 4 & 4 \end{bmatrix}$$

$$|AB| = 12(72-40) - 20(60-40) + 6(60-72)$$

$$= 12(32) - 20(20) + 6(-12)$$

$$= 384 - 400 - 72$$

$$= 384 - 472$$

$$\boxed{|AB| = -88} \longrightarrow ①$$

$$|A| = 2(4-2) - 3(1-0) + 1(1-0)$$

$$= 2(2) - 3(1) + 1(1)$$

$$= 4 - 3 + 1$$

$$= 5 - 3$$

$$\boxed{|A| = 2}$$

$|B| = 1(6-2) - 6(9-1) + 0(6-2)$
$= 1(4) - 6(8) + 0(4)$
$= 4 - 48 + 0$
$= -44$

$|A|\cdot|B| = 2 \times -44$
$= -88 \longrightarrow ②$

from eq ① & ②,

$$\boxed{|AB| = |A|\cdot|B|}$$

verified

② $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{bmatrix}$.

Sol Given matrices

$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3+2 & 2+12+3 & 1+6+1 \\ 2+1+2 & 4+4+3 & 2+2+1 \\ 4+2+2 & 8+8+3 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 17 & 8 \\ 5 & 11 & 5 \\ 8 & 19 & 9 \end{bmatrix}$$

$|AB| = 6(99-95) - 17(45-40) + 8(95-88)$
$= 6(4) - 17(5) + 8(7)$
$= 24 - 85 + 56$
$= 80 - 85$
$= -5 \longrightarrow ①$

$|A| = 1(1-2) - 3(2-4) + 1(4-4)$
$= 1(-1) - 3(-2) + 1(0)$
$= -1 + 6$
$= 5$

$|B| = 1(4-6) - 2(1-4) + 1(3-8)$
$= 1(-2) - 2(-3) + 1(-5)$

$$= -2+6-5$$
$$= -7+6$$
$$= -1$$

$$|A|\cdot|B| = 5\times(-1)$$
$$= -5 \rightarrow (2)$$

from eq (1) & (2)

$$|AB| = |A|\cdot|B|$$

verified

(3) $A=\begin{bmatrix}1 & 2 & 1\\ 3 & 5 & 6\\ 8 & 7 & -5\end{bmatrix}$ and $B=\begin{bmatrix}7 & 2 & 3\\ 5 & -4 & -2\\ 6 & -3 & 1\end{bmatrix}$

sol Given matrices

$$A=\begin{bmatrix}1 & 2 & 1\\ 3 & 5 & 6\\ 8 & 7 & -5\end{bmatrix} \text{ and } B=\begin{bmatrix}7 & 2 & 3\\ 5 & -4 & -2\\ 6 & -3 & 1\end{bmatrix}$$

$$AB = \begin{bmatrix}1 & 2 & 1\\ 3 & 5 & 6\\ 8 & 7 & -5\end{bmatrix}\begin{bmatrix}7 & 2 & 3\\ 5 & -4 & -2\\ 6 & -3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}7+10+6 & 2+(-8)-3 & 3+(-4)+1\\ 21+25+36 & 6+(-20)-18 & 9-10+6\\ 56+35-30 & 16-28+15 & 24-14-5\end{bmatrix}$$

$$= \begin{bmatrix}23 & -9 & 0\\ 82 & -32 & 5\\ 61 & 3 & 5\end{bmatrix}$$

$$|AB| = 23(-160-15)+9(410-305)+0(246+1952)$$
$$= 23(-175)+9(105)+0(2198)$$
$$= -4025+945$$
$$= -3080 \rightarrow (1)$$

$$|A| = 1(-25-42)-2(-15-48)+1(21-40)$$
$$= 1(-67)-2(-63)+1(-19)$$
$$= -67+126-19$$
$$= \cancel{-222}\ 40$$

$$|B| = 7(-4-6)-2(5+12)+3(-15+24)$$
$$= 7(-10)-2(17)+3(9)$$
$$= -70-34+27$$

$$= -104 + 27$$

$$= -77$$

$$|A| \cdot |B| = -212 \times -17$$

$$= 3080$$

$$|AB| = |A| \cdot |B|$$

verified

Inverse of a matrix by using Gauss Jordan Method

Inverse of non-singular matrix by using Gauss Jordan Method

10M

Consider a matrix 'A'

step 1:- Let $A = I_n A$, then.

step 2:- Convert LHS matrix into identity matrix by using elementary row operations and then simultaneously apply the same operations on RHS unit matrix, we get $I = BA$.

step 3:- We get $I = BA$ where B is the inverse of the given matrix i.e $\boxed{B = A^{-1}}$

1. find the inverse of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ by using gauss jordan method

sol

Given matrices

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Let $A = I_3 A$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \rightarrow R_3 - 2R_1$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

$R_3 \rightarrow 3R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix} A$$

$R_1 \rightarrow 3R_1 + 2R_2$

$$\begin{bmatrix} 3 & 0 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix} A.$$

$R_1 \rightarrow 5R_1 + 2R_3$

$$\begin{bmatrix} 15 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} -9 & 6 & 6 \\ -2 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{15}$

$R_3 \rightarrow R_3/-5$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9/15 & 6/15 & 6/15 \\ -2 & 1 & 0 \\ +2/5 & 2/5 & -3/5 \end{bmatrix} A$$

$R_2 \rightarrow R_2 + 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ -6/5 & 9/5 & -6/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1/5 \begin{bmatrix} -3 & 2 & 2 \\ -6 & 9 & -6 \\ 2 & 2 & -3 \end{bmatrix} A$$

$R_2 \rightarrow R_2 \times (-1/3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1/5 \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} A$$

$I_3 = BA$

where $B = A^{-1}$

$$A^{-1} = 1/5 \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$ //

2. find the inverse of $A=\begin{bmatrix}1 & 6 & 4\\0 & 2 & 3\\0 & 1 & 2\end{bmatrix}$ by using gauss jordan method.

sol Given

$$A=\begin{bmatrix}1 & 6 & 4\\0 & 2 & 3\\0 & 1 & 2\end{bmatrix}$$

Let $A = I_3 A$

$$\begin{bmatrix}1 & 6 & 4\\0 & 2 & 3\\0 & 1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A$$

$R_3 \to 2R_3 - R_2$

$$\begin{bmatrix}1 & 6 & 4\\0 & 2 & 3\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 2\end{bmatrix} A$$

$R_1 \to R_1 - 3R_2$

$$\Rightarrow \begin{bmatrix}1 & 0 & -5\\0 & 2 & 3\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -3 & 0\\0 & 1 & 0\\0 & -1 & 2\end{bmatrix} A$$

$R_2 \to R_2 - 3R_3$

$$\begin{bmatrix}1 & 0 & -5\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -3 & 0\\0 & 4 & -6\\0 & -1 & 2\end{bmatrix} A$$

$R_1 \to R_1 + 5R_3$

$R_2 \to R_2/2$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -8 & 10\\0 & 2 & -3\\0 & -1 & 2\end{bmatrix} A$$

$I_3 = BA$

where $B = A^{-1}$

$$\therefore A^{-1} = \begin{bmatrix}1 & -8 & 10\\0 & 2 & -3\\0 & -1 & 2\end{bmatrix}$$

3) find the inverse of $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ by using Gauss jordon method.

sol Given

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Let $A = I_3 A$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \longrightarrow 2R_3 + R_1$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} A$$

$R_3 \rightarrow R_3 + 5R_2$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -0 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_1 \rightarrow R_1 + R_2$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_2 \rightarrow 8R_2 - R_3$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 4 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_1 \rightarrow 2R_1 - R_3$

$$\Rightarrow \begin{bmatrix} -4 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_1 \to R_1/4$, $R_2 \to \frac{R_2}{-8}$, $R_3 \to \frac{R_3}{8}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & 2/4 \\ 1/8 & -3/8 & 2/8 \\ 1/8 & 5/8 & 2/8 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1/8 \begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix} A$$

$$I_3 = BA$$

where $B = A^{-1}$

$$\therefore A^{-1} = \frac{1}{8} \begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

(4) find the inverse of $A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by using Gauss jordon method.

sol Given

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Let $A = I_3 A$.

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2/2$

$R_3 \to R_3/3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$\therefore I_3 = BA$

$B = A^{-1}$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

⑤ find the inverse of $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}$ by using Gauss Jordon method.

Sol Given :

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}$$

Let $A = I_3 A$

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -5 & 3 \\ 1 & -2 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -5 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

$R_2 \to R_2 \times (-1)$

$R_3 \to R_3 \times (-1)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -3 \\ -1 & 2 & -1 \end{bmatrix}$$

$I_3 = BA$

where $B = A^{-1}$

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -3 \\ -1 & 2 & -1 \end{bmatrix}$$

⑥ find the inverse of $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ by using Gauss Jordon method.

sol Given

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Let $A = I_3 A$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - R_3 \,,\; R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_3 \,,\; R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix} A$$

$R_1 \to R_1 + 2R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix} A$$

$R_1 \to R_1/2 \,,\; R_2 \to R_2/3 \,,\; R_3 \to R_3/-2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

$$I_3 = BA$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ by using gauss gorder method

sol Given A

Eg:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

$R_2 \rightarrow 3R_1 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \rightarrow 2R_1 + R_2$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \rightarrow 2R_1 - R_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -4 \\ -3 & 1 & 0 \\ -1 & -1 & 4 \end{bmatrix} A$$

$R_1 \rightarrow R_1/4$; $R_2 \rightarrow \frac{R_2}{-4}$; $R_3 \rightarrow R_3/4$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

$I = BA$

$$A^{-1} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} //$$

(8) find the inverse of the $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ by using gauss jordon method.

imp

P

sol Given

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Let $A = I_4 A$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + 2R_1$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to 2R_3 - 11R_2$

$R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 2R_1 - 3R_2$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to 3R_2 + R_3$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -6 & 0 & -2 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -4 & -8 & 2 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 6R_4$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -6 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -4 & -8 & 2 & 0 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -6 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -4 & -8 & 2 & 0 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -6 & -6 & 6 & 12 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1/-2, \quad R_2 \to R_2/-6$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$$I_4 = BA$$

where $B = A^{-1}$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

(8) find the inverse of the $A=\begin{bmatrix}2 & -6 & -2 & -3\\ 5 & -13 & -4 & -7\\ -1 & 4 & 1 & 2\\ 0 & 1 & 0 & 1\end{bmatrix}$ by using gauss jordan method.

Sol Given $A=\begin{bmatrix}2 & -6 & -2 & -3\\ 5 & -13 & -4 & -7\\ -1 & 4 & 1 & 2\\ 0 & 1 & 0 & 1\end{bmatrix}$

Let $A = I_4 A$

$$\begin{bmatrix}2 & -6 & -2 & -3\\ 5 & -13 & -4 & -7\\ -1 & 4 & 1 & 2\\ 0 & 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} A$$

$R_3 \to 2R_3 + R_1$

$R_2 \to 2R_2 - 5R_1$

$$\begin{bmatrix}2 & -6 & -2 & -3\\ 0 & 4 & 2 & 1\\ 0 & 2 & 0 & 1\\ 0 & 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ -5 & 2 & 0 & 0\\ +1 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} A$$

$R_4 \to 2R_4 - R_3$

$$\begin{bmatrix}2 & -6 & -2 & -3\\ 0 & 4 & 2 & 1\\ 0 & 2 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ -5 & 2 & 0 & 0\\ 1 & 0 & 2 & 0\\ -1 & 0 & -2 & 2\end{bmatrix} A$$

$R_3 \to 2R_3 - R_2$

$$\begin{bmatrix}2 & -6 & -2 & -3\\ 0 & 4 & 2 & 1\\ 0 & 0 & -2 & +1\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ -5 & 2 & 0 & 0\\ 7 & -2 & 4 & 0\\ -1 & 0 & -2 & 2\end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix}2 & -6 & -2 & -3\\ 0 & 4 & 0 & 2\\ 0 & 0 & -2 & 1\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 0 & 4 & 0\\ 7 & -2 & 4 & 0\\ -1 & 0 & -2 & 2\end{bmatrix} A$$

$R_3 \to R_3 - R_4$

$$\begin{bmatrix} 2 & -6 & -2 & -3 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 8 & -2 & 6 & -2 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 2 & -6 & 0 & -3 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 & 6 & -2 \\ 2 & 0 & 4 & 0 \\ 8 & -2 & 6 & -2 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$R_1 \to 2R_1 + 3R_2$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 24 & -2 & 24 & -4 \\ 2 & 0 & 4 & 0 \\ 8 & -2 & 6 & -2 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_4$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 24 & -2 & 24 & -4 \\ 4 & 0 & 8 & -4 \\ 8 & -2 & 6 & -2 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$R_1 \to R_1/4, \; R_2 \to R_2/4$

$R_3 \to R_3/-2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1/2 & 6 & -1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix} A$$

$I_4 = BA$

where $B = A^{-1}$

$$A^{-1} = \begin{bmatrix} 6 & -1/2 & 6 & -1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \\ -1 & 0 & -2 & 2 \end{bmatrix}$$

//

10. find the inverse of the $A=\begin{bmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 2 & 3\\ 2 & 2 & 2 & 3\\ 2 & 3 & 3 & 3\end{bmatrix}$ by using gauss jordon method

Sol Given

$$A=\begin{bmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 2 & 3\\ 2 & 2 & 2 & 3\\ 2 & 3 & 3 & 3\end{bmatrix}$$

Let $A = I_4 A$

$$\begin{bmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 2 & 3\\ 2 & 2 & 2 & 3\\ 2 & 3 & 3 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} A$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 2 & 3\\ 2 & 2 & 2 & 3\\ 0 & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix} A$$

$R_3 \to R_2 - 2R_2$

$$\begin{bmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 2 & 3\\ 0 & 0 & -2 & -3\\ 0 & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 0 & 0\\ 0 & 0 & -2 & -3\\ 0 & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix} A$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\\ 1 & -2 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix} A$$

$R_1 \to R_1 - R_4$

$$\begin{bmatrix}0 & -1 & 0 & 2\\ 1 & 1 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & -2\\ 0 & -1 & 1 & 0\\ 1 & -2 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix} A$$

$R_1 \to R_1 + 2R_3$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 & -2 \\ 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 & -2 \\ -3 & 3 & -3 & 2 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 & -2 \\ -3 & 3 & -3 & 2 \\ -2 & 2 & -3 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 & -2 \\ -3 & 3 & -3 & 2 \\ -2 & 2 & -3 & 2 \\ -3 & 3 & -5 & 3 \end{bmatrix}$$

$R_3 \to R_3/-1$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 & -2 \\ -3 & 3 & -3 & 2 \\ 2 & -2 & 3 & -2 \\ -3 & 3 & -5 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_2, \; R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 & -2 \\ -3 & 3 & -3 & 2 \\ 2 & -2 & 3 & -2 \\ -3 & 3 & -5 & 3 \end{bmatrix}$$

$I_4 = BA$

where $B = A^{-1}$

$$A^{-1} = \begin{bmatrix} 3 & -4 & 4 & -2 \\ -3 & 3 & -3 & 2 \\ 2 & -2 & 3 & -2 \\ -3 & 3 & -5 & 3 \end{bmatrix}$$

System of linear equations

Let us consider a system of 'm' linear equations in 'n' annos say $x_1, x_2, x_3 ---- x_n$ as below

$$a_{11}x_1 + a_{12}x_2 + ----- + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ----- + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ---- + a_{mn}x_n = b_n$$

where 'a' and 'b' are constants it may be real or complex.

The system of above equations can be written in the matrix form

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & ---- & a_{1n} \\ a_{21} & a_{22} & ---- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & ---- & a_{mn} \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{B}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}} \longrightarrow ①$$

where coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & ---- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & ---- & a_{mn} \end{bmatrix}$

'X' is variable matrix and B is constant matrix.

Homogeneous And Non-Homogeneous System of Equations

* If $B = 0$ in equation ① then the system is homogeneous. Otherwise the system is said to be non-homogeneous.
* The system $AX = B$ is consistant that is it has a solution if and only if rant of A is equal to the rank of ao augmanted matrix $[\rho(AB)]$.
* The system $AX=0$ is always consistant since $X=0$ is always a solution of $AX=0$. This solution is called

trivial solution.

Working Rule (or) Homogeneous :-

* Let the rank of $A = r$.

If $r = n$ then the system of equations have trivial solution that is [0 solution].

* If $r < n$ then the system of equations have non tr infinite number of solutions that is non trival solution. In this case we have $(n-r)$ linearly independent solutions.

Problems

1. Solve completely the system of equations

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

Sol Given equations

$$\left.\begin{aligned} x + 3y - 2z &= 0 \\ 2x - y + 4z &= 0 \\ x - 11y + 14z &= 0 \end{aligned}\right\} \rightarrow (1)$$

eqn (1) can be written in matrix form

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow ②$$

no. of unknowns $n = 3$

$\rho(A) = r = 2$

$\therefore$ $r < n$, It has infinite no. of solutions

In this case we have

$(n-r)$ independent solutions

$\Rightarrow n - r = 3 - 2 = 1$

Let $\boxed{z = c}$

from ②

$x + 3y - 2z = 0 \rightarrow ③$

$-7y + 8z = 0 \rightarrow ④$

from ④

$-7y + 8c = 0$

$-7y = -8c$

$y = 8/7\, c$

from ③

$x + 3(8/7)c - 2c = 0$

$x + \frac{24}{7}c - 2c = 0$

$x + \frac{10}{7}c = 0$

$x = \frac{-10}{7}c$

Hence, the solution is

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix} c$$

② Solve completely the system of equations

$$\left.\begin{aligned} x + 2y + 3z &= 0 \\ 3x + 4y + 4z &= 0 \\ 7x + 10y - 12z &= 0 \end{aligned}\right\} \rightarrow ①$$

sol Given equations

$$\left.\begin{array}{l} x+2y+3z=0 \\ 3x+4y+4z=0 \\ 7x+10y-12z=0 \end{array}\right\} \rightarrow ①$$

eqn ① can be written in matrix form

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & -12 \end{bmatrix}$

$R_3 \rightarrow R_3 - 7R_1$
$R_2 \rightarrow R_2 - 3R_1$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -33 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & -23 \end{bmatrix}$$

no. of unknowns $n=3$

$\rho(A) = r = 3$

$\therefore n=r$, then the system has trivial solutions.

③ Solve completely the system of equations

$$4x+2y+z+3w=0$$
$$6x+3y+4z+7w=0$$
$$2x+y+w=0$$

sol Given equations

$$\left.\begin{array}{l} 4x+2y+z+3w=0 \\ 6x+3y+4z+7w=0 \\ 2x+y+w=0 \end{array}\right\} \rightarrow ①$$

eqn ① can be written in Matrix form

$$\underbrace{\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{X} = 0$$

Now
$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - R_1$

$R_2 \to 4R_2 - 6R_1$

$$= \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_3 \to 10R_3 + R_2$

$$= \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2/10$

$$= \begin{bmatrix} 4 & 2 & \cancel{1} & \cancel{3} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to ②$$

$\rho(A) = r = 2$

no. of unknowns, $n = 4$

$\therefore r < n$, then the system has infinite no of non trivial solutions

In this case, we have $(n-r)$ independent solutions

$n - r = 4 - 2 = 2$

Let $\boxed{x = c_1}$ and $\boxed{w = c_2}$

from ②

$$4x + 2y + z + 3w = 0$$
$$z + 3w = 0$$
$$z + 3c_2 = 0$$
$$\boxed{z = -c_2}$$

$$4c_1 + 2y - c_2 + 3c_2 = 0$$
$$2y + 4c_1 + 2c_2 = 0$$
$$2y = -2(2c_1 + c_2)$$
$$\boxed{y = -2c_1 - c_2}$$

$z + 3w = 0$, $z = -3w$, $z = -3c_2$

$z + 3w = 0$, $z = -3w$, $z = -3c_2$

$$\therefore X = \begin{array}{cc} \begin{bmatrix} c_1 \\ -2c_1 - c_2 \\ -c_2 \\ c_2 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \\ w \end{array} \end{array}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} c_2$$

(4) Solve completely the system of equations

$$\left.\begin{aligned} x_1 + 2x_3 - 2x_4 &= 0 \\ 2x_1 + x_2 - x_4 &= 0 \\ 4x_1 - x_2 + 3x_3 + x_4 &= 0 \end{aligned}\right\} \rightarrow (1)$$

Sol Given equations

$$\left.\begin{aligned} x_1 + 2x_3 - 2x_4 &= 0 \\ 2x_1 + x_2 - x_4 &= 0 \\ 4x_1 - x_2 + 3x_3 + x_4 &= 0 \end{aligned}\right\} \rightarrow (1)$$

eqn (1) can be written in Matrix form

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 4 & -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Now

$$A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 4 & -1 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 4R_1$

$R_2 \to R_2 - 2R_1$

$$= \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & -1 & -5 & 9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & -1 & 6 \end{bmatrix} \to (2)$$

$\rho(A) = 3$

no. of unknowns $n = 4$

$\therefore$ $r < n$, the system has infinite no. of non trivial solutions

so, we have $(n-r)$ independent solutions

$n - r = 4 - 3 = 1$

Let us take $\boxed{x_4 = c}$

from (2)

$x_1 + 2x_3 - 2x_4 = 0 \to (3)$

$-x_2 - 4x_3 + 3x_4 = 0 \to (4)$

$-x_3 + 6x_4 = 0$

$-x_3 + 6c = 0$

$-x_3 = -6c$

$\boxed{x_3 = 6c}$

from (4)

$-x_2 - 4(6c) + 3c = 0$

$-x_2 - 24c + 3c = 0$

$-x_2 - 21c = 0$

$-x_2 = 21c$

$\boxed{x_2 = -21c}$

from (3)

$x_1 + 2(-21c) - 2c = 0$

$$x_1 - 42c - 2c = 0$$
$$x_1 - 44c = 0$$
$$\boxed{x_1 = 44c}$$

Hence,

$$\therefore X = \begin{bmatrix} 44 \\ -21 \\ 6 \\ 1 \end{bmatrix} c$$

which is the required solution for the given system of equation

(5) show that the only real number λ for which the system $x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$, $2x+3y+z=\lambda z$ has non zero solution is 6 and solve them when $\lambda = 6$.

Sol Given equations

$$\cancel{(1-\lambda)x+2y+3z=\lambda x}$$
$$\cancel{3x+y+2z=\lambda y} \quad \to (1)$$
$$\cancel{2x+3y+z=\lambda z}$$

$$\left.\begin{aligned} (1-\lambda)x+2y+3z &= 0 \\ 3x+(1-\lambda)y+2z &= 0 \\ 2x+3y+(1-\lambda)z &= 0 \end{aligned}\right\} \to (1)$$

eqn (1) cas be written in matrix form

$$\underbrace{\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = 0$$

Now

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$$\rho(A) < 3$$

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$R_1 \to R_1+R_2+R_3$

$$= \begin{bmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$$(6-\lambda)\begin{bmatrix} 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$

$$(6-\lambda)\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{bmatrix}$$

$$= (6-\lambda)\left[1\left[+(2+\lambda)(1+\lambda)+1\right]-0+0\right]$$

$$= (6-\lambda)\left[2+2\lambda+\lambda+\lambda^2+1\right]$$

$$\Rightarrow (6-\lambda)(\lambda^2+3\lambda+3)=0$$

$$6-\lambda=0$$

$$6=\lambda$$

$$\boxed{\lambda=6}$$

Given that the system of equation has non zero solution we get $\lambda=6$ is the only real number and other complex. When $\lambda=6$ the given system becomes

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & 2 & 3 \\ 3 & 1-6 & 2 \\ 2 & 3 & 1-6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}$$

$R_3 \to 5R_3+2R_1$

$R_2 \to 5R_2+3R_1$

$$= \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}$$

$R_3 \to R_3+R_2$

$$= \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2/19$

$$= \begin{bmatrix} -5 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to ②$$

$\rho(A) = r = 2$

no.of unknowns $n = 3$

$\therefore$ $r < n$, the system has infinite no.of non trivial solutions.

so, we have $n-r$ independent solutions

$\Rightarrow n - r = 3 - 2 = 1$

Let $\boxed{z = k}$

from ②

$$-5x + 2y + 3z = 0 \rightarrow ③$$

$$-y + z = 0 \rightarrow ④$$

from ④

$$-y + k = 0$$

$$-y = -k$$

$$\boxed{y = k}$$

from ③

$$-5x + 2k + 3k = 0$$

$$-5x + 5k = 0$$

$$-\cancel{5}x = -\cancel{5}k$$

$$\boxed{x = k}$$

$x + y - 3z + 2w = 0$
$2x - y + 2z - 3w = 0$
$3x - y + z - 4w = 0$
$-4x + 5y - 3z + w = 0$

Hence, the required solution is

$x = k$, $y = k$ and $z = k$

⑥ Solve completely the system of equations

$$x + y - 3z + 2w = 0$$
$$2x - y + 2z - 3w = 0$$
$$3x - y + z - 4w = 0$$
$$-4x + 5y - 3z + w = 0$$

Sol Given equations

$$\left.\begin{aligned} x + y - 3z + 2w &= 0 \\ 2x - y + 2z - 3w &= 0 \\ 3x - y + z - 4w &= 0 \\ -4x + 5y - 3z + w &= 0 \end{aligned}\right\} \rightarrow ①$$

eq ① can be written in matrix form

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -1 & 1 & -4 \\ -4 & 5 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -1 & 1 & -4 \\ -4 & 5 & -3 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 + 4R_1$

$$= \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & -7 & -7 \\ 0 & -4 & 10 & -10 \\ 0 & 9 & 9 & 9 \end{bmatrix}$$

$R_4 \to 3R_4 + 9R_2$

$$= \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & -7 & -7 \\ 0 & -4 & 10 & -10 \\ 0 & 0 & 36 & 36 \end{bmatrix}$$

$R_3 \to 3R_3 - 4R_2$

$$= \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & -7 & -7 \\ 0 & 0 & 58 & -2 \\ 0 & 0 & 36 & 36 \end{bmatrix}$$

$R_4 \to R_4/36$

$$= \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & -7 & -7 \\ 0 & 0 & 58 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

no. of unknown rows $n = 4$

$\rho(A) = 4$

$\therefore n = r$ then the system has trivial solutions.

Working rule of Non-Homogeneous

If rank of A =

i) If $\rho(A) = \rho(A:B) = n$ then the system has unique solution(or) trivial solution.

ii) If $\rho(A:B) \neq \rho(A)$ then the system has no solution.

iii) If $\rho(A:B) = \rho(A) < n$ then the system is consistente but their existiate no.of solutions (or) infinite no.of solutions. In this case we obtain a $n-r$ variables any orbitary values and solve for the remaining unknown values.

Problems:

1) Show that the equations
$$x+y+z=4$$
$$2x+5y-2z=3$$
$$x+7y-7z=5$$ are not consistant

sol Given that
$$\left.\begin{aligned} x+y+z&=4 \\ 2x+5y-2z&=3 \\ x+7y-7z&=5 \end{aligned}\right\} \rightarrow (1)$$

eqn (1) can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$
$$A \qquad X = B$$

consider, a Augmented matrix.

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 3 & -4 & : & 5 \\ 0 & 0 & 0 & : & 11 \end{bmatrix}$$

$\rho(A:B) = 3$

$\rho(A) = 2$

$\rho(A:B) \neq \rho(A)$

Hence the system has not consistant

2 find whether the following equations are consistant

$$\left.\begin{aligned} x+y+2z &= 4 \\ 2x-y+3z &= 9 \\ 3x-y-z &= 2 \end{aligned}\right\}$$

Sol Given equations

$$\left.\begin{aligned} x+y+2z &= 4 \\ 2x-y+3z &= 9 \\ 3x-y-z &= 2 \end{aligned}\right\} \to ①$$

eqn ① can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

$A \quad X = B$

Consider, a Augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

$R_3 \to 3R_3 - 4R_2$

$$= \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

$R_3 \to \frac{R_3}{-17}$

$$= \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to (2)$$

$\rho(A:B) = 3$

$\rho(A) = 3$

$\rho(A:B) = \rho(A) = n$

Hence, the system is unique solution.

from (2)

$\boxed{z = 2}$

$-3y - z = 1$

$-3y - 2 = 1$

$-3y = 1+2$

$-3y = 3$

$\therefore \boxed{y = -1}$

$x + y + 2z = 4$

$x - 1 + 2 \cdot 2 = 4$

$x - 1 + 4 = 4$

$x + 3 = 4$

$x = 4 - 3$

$\boxed{x = 1}$

from (2)

$x + y + 2z = 4 \to (3)$

$-3y - z = 1 \to (4)$

$\boxed{z = 2} \to (5)$

fr

$\therefore$ The required solution of given system is $x = 1$, $y = -1$ and $z = 2$ //

find whether the given equation are consistant or non consistant

$$x+y+z=9$$
$$2x+5y+7z=52$$
$$2x+y-z=0$$

Given equations

$$\left.\begin{array}{l} x+y+z=9 \\ 2x+5y+7z=52 \\ 2x+y-z=0 \end{array}\right\} \rightarrow (1)$$

eqn (1) can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$A \quad X = B$$

consider, a Augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 3 & 5 & : & 34 \\ 0 & 0 & -4 & : & -20 \end{bmatrix} \rightarrow (2)$$

$\rho(A:B) = 3$

$\rho(A) = 3$

$\rho(A:B) = \rho(A) = n$

Hence, the system is unique solution

Hence, the given system has unique solution

from ② ← $x+y+z=9$

$3y+5z=34$

$-4z=-20$

$\boxed{z=5}$

$\boxed{z=5}$

$3y+5z=34$

$3y+5(5)=34$

$3y=34-25$

$3y=9$

$y=\frac{9}{3}$

$\therefore \boxed{y=3}$

$x+y+z=9$

$x+5+3=9$

$x+8=9$

$x=9-8$

$\therefore \boxed{x=1}$

$\therefore$ The required solution of given system

$\boxed{x=1}$ $\boxed{y=3}$ $\boxed{z=5}$

④ find whether the given equation are consistant or non consistant.

$x+2y+3z=1$

$2x+3y+8z=2$

$x+y+z=3$

sol Given equations

$$\left.\begin{cases} x+2y+3z=1 \\ 2x+3y+8z=2 \\ x+y+z=3 \end{cases}\right\} \rightarrow ①$$

eq ① can be written in matrix form

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

consider, a Augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 8 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{bmatrix} \to (2)$$

from (2)

$x + 2y + 3z = 1$

$-y + 2z = 0$

$-4z = 2$

$-2z = 1$

$$\boxed{z = \frac{-1}{2}}$$

$-y + 2\left(\frac{-1}{2}\right) = 0$

$-y - 1 = 0$

$-y = 1$

$$\boxed{y = -1}$$

$x + 2(-1) + 3\left(\frac{-1}{2}\right) = 1$

$x - 2 + 3\left(\frac{-1}{2}\right) = 1$

$2x - 4 - 3 = 2$

$2x - 7 = 2$

$2x = 2 + 7$

$2x = 9$

$$\boxed{x = \frac{9}{2}}$$

Gauss Elimination method:-

This method of solving a system of m linear equations in n unknowns consist eliminating co-efficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution.

Problems:-

1. Solve the equations $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$ by using Gauss elimination method

G.T
$$\left.\begin{aligned} 3x+y+2z &= 3 \\ 2x-3y-z &= -3 \\ x+2y+z &= 4 \end{aligned}\right\} \rightarrow (1)$$

Sol Given eqns
$$\left.\begin{aligned} 3x+y+2z &= 3 \\ 2x-3y-z &= -3 \\ x+2y+z &= 4 \end{aligned}\right\} \rightarrow (1)$$

eqn (1) can be written in the matrix form

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$A \quad X = B$$

Now, we have to reduce in upper triangular matrix

The Augmented matrix is.

$$[A:B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 - R_1$

$R_2 \rightarrow 3R_2 - 2R_1$

$$\sim \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -11 & -7 & -15 \\ 0 & 5 & 1 & 9 \end{bmatrix}$$

$R_3 \rightarrow 11R_3 + 5R_2$

$$= \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -11 & -7 & -15 \\ 0 & 0 & -24 & 24 \end{bmatrix}$$

$$R_3 \to R_3 / 24$$

$$\begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -11 & -7 & -15 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to (2)$$

from (2)

using backword substitution

$$-z = 1$$

$$\boxed{z = -1}$$

$$-11y - 7z = -15$$

$$-11y - 7(-1) = -15$$

$$-11y + 7 = -15$$

$$-11y = -15 - 7$$

$$-\cancel{11}y = -\cancel{22}^{2}$$

$$\boxed{y = 2}$$

$$3x + y + 2z = 3$$

$$3x + 2 + 2(-1) = 3$$

$$3x = 3$$

$$\boxed{x = 1}$$

Hence the required solution

$x = 1$, $y = 2$, & $x = -1$

(2) Solve the following equations $3x + y - z = 3$, $2x - 8y + z = -5$ $x - 2y + 9z = 8$ by using Gauss elimination method.

sol Given eqns

$$\left.\begin{aligned} 3x + y - z &= 3 \\ 2x - 8y + z &= -5 \\ x - 2y + 9z &= 8 \end{aligned}\right\} \to (1)$$

eqn(1) can be written in the matrix form

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$$

Now we have to reduces into upper triangular matrix

$$[A:B] = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$= \begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 0 & 4 & 17 & 21 \end{bmatrix}$$

$R_3 \to 13R_3 + 2R_2$

$$\begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & 0 & 231 & 231 \end{bmatrix}$$

$R_3 \to \frac{R_3}{231}$

$$\begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26 & 5 & -21 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to (2)$$

from (2)

using Backward substitution

$\boxed{z=1}$

$-26y + 5z = -21$

$-26y + 5(1) = -21$

$-26y = -21-5$

$y = \frac{-26}{-26}$

$\boxed{y=1}$

$3x + y - z = 3$

$3x + 1 - 1 = 3$

$3x = 3$

$\boxed{x=1}$

Hence, the required solution is $x=1, y=1, z=1$.

3. $x_1 - x_2 + x_3 = 1$

$-3x_1 + 2x_2 - 3x_3 = -6$

$2x_1 - 5x_2 + 4x_3 = 5$

eq (1) can be written in the matrix form

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix}$$

$$A \quad X = B$$

We have to reduce into upper triangular matrix.

Now, The Augmented matrix

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & 6 \\ 2 & -5 & 4 & 5 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1$$

$$R_2 \to R_2 + 3R_1$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{bmatrix} \to (2)$$

from (2)

Using Backward substitution

$$2x_3 = 12$$

$$\boxed{x_3 = 6}$$

$$-x_2 = -3$$

$$\boxed{x_2 = 3}$$

$$x_1 - x_2 + x_3 = 1$$

$$x_1 - 3 + 6 = 1$$

$$x_1 = 1 - 3$$

$$\boxed{x_1 = -2}$$

∴ Hence the required solution is $x_1 = -2$, $x_2 = 3$, $x_3 = 6$

(4)

$$\left.\begin{aligned} x + 3y + 2z &= 5 \\ 2x + 4y - 6z &= -4 \\ x + 5y + 3z &= 10 \end{aligned}\right\} \to (1)$$

eq (1) can be written in the matrix form

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 10 \end{bmatrix}$$

$$A \quad X \quad B$$

We have to reduce into upper triangular matrix

$$[A:B] = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 2 & 1 & 5 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 0 & -9 & -9 \end{bmatrix}$$

$R_3 \rightarrow \frac{R_3}{-9}$

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow (2)$$

from (2)

Using Backward substitution

$\boxed{z = 1}$

$-2y - 10z = -14$

$-2y - 10(1) = -14$

$-2y = -14 + 10$

$+2y = \cancel{} + 4$

$\boxed{y = 2}$

$x + 3y + 2z = 5$

$x + 3(2) + 2(1) = 5$

$x + 6 + 2 = 5$

$x = 5 - 8$

$\boxed{x = -3}$

Hence the required solution is

$x = -3, \; y = 2, \; z = 1$ //.

(5) Express the following system in matrix form and solve gauss elimination.

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$
$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_3 + 3x_2 + 3x_3 - 3x_4 = -1$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10$$

Sol Given equations

$$\left.\begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 6x_1 - 6x_2 + 6x_3 + 12x_4 &= 36 \\ 4x_3 + 3x_2 + 3x_3 - 3x_4 &= -1 \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10 \end{aligned}\right\} \rightarrow (1)$$

eqn (1) can be written in the matrix form

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

The Augmented matrix is

$$[A:B] = \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$R_2 \rightarrow R_2/6$

$$= \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & -3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$= \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 2R_1$

$R_3 \rightarrow R_3 - 4R_1$

$R_2 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -2 \\ 0 & 4 & -3 & -3 & -2 \end{bmatrix}$$

$R_4 \rightarrow 3R_4 - 4R_2$

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 0 & -9 & 3 & 18 \end{bmatrix}$$

$R_2 \rightarrow \frac{R_2}{3}, \; R_4 \rightarrow \frac{R_4}{3}$

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 0 & -3 & 1 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 7R_2$

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & -3 & 1 & 6 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 3R_3$

$$\begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & 0 & 13 & 39 \end{bmatrix} \longrightarrow (2)$$

from (2)

By using Backward substitution

$13x_4 = 39$

$$\boxed{x_4 = 3}$$

$-x_3 - 4x_4 = -11$

$-x_3 - 4(3) = -11$

$$-x_3 = -11+12$$

$$-x_3 = 1$$

$$\boxed{x_3 = -1}$$

$$x_2 - x_4 = -2$$

$$x_2 - 3 = -2$$

$$x_2 = -2+3$$

$$\boxed{x_2 = 1}$$

$$x_1 - x_2 + x_3 + 2x_4 = 6$$

$$x_1 - 1 - 1 + 2(3) = 6$$

$$x_1 + 4 = 6$$

$$x_1 = 6-4$$

$$\boxed{x_1 = 2}$$

Gauss. Jacobi method (or) Jacobi Interative method

Consider the system of equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \rightarrow ①$$

If a_{11}, a_{22} & a_{33} are large as compared to other coefficients (that is, it must satisfy diagonal dominance property). then solving these for x, y & z respectively. Then the system ① can be written as $\Rightarrow$

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}[b_1 - a_{12}y - a_{13}z] \\ y &= \frac{1}{a_{22}}[b_2 - a_{21}x - a_{23}z] \\ z &= \frac{1}{a_{33}}[b_3 - a_{31}x - a_{32}y] \end{aligned}\right\} \rightarrow ②$$

Let us start with initial approximations

$x^{(0)}, y^{(0)}$ & $z^{(0)}$ (i.e; $x=0, y=0, z=0$)

for the values of x, y & z.

Substitute in these values in eq ② we get 1st approximation $x^{(1)}, y^{(1)}$ & $z^{(1)}$.

Substitute 1st approximation values in eq ② we get 2nd approximation $x^{(2)}, y^{(2)}$ & $z^{(2)}$.

The procedure is continued till the two successive approximations are equal.

<u>Note</u>:- If matrix is not diagonal dominance then it try to convert into diagonal dominance by rearranging system of equations.

Problems

1. Solve by Jacobian interation method for the equations are

$$4x + 2y + z = 14$$
$$x + 5y - z = 10$$
$$x + y + 8z = 20$$

sol Given eqns

$$\left.\begin{aligned} 4x + 2y + z &= 14 \\ x + 5y - z &= 10 \\ x + y + 8z &= 20 \end{aligned}\right\} \rightarrow (1)$$

eqn (1) can be written in the form

$$4x = 14 - 2y - z$$
$$x = \frac{1}{4}[14 - 2y - z]$$
$$5y = 10 - x + z$$
$$y = \frac{1}{5}[10 - x + z]$$
$$z = \frac{1}{8}[20 - x - y]$$

1st iteration

Let us take initial values of x, y & z is $x^{(0)} = 0$, $y^{(0)} = 0$ & $z^{(0)} = 0$

$$x^{(1)} = \frac{1}{4}[14 - 2y^{(0)} - z^{(0)}]$$
$$= \frac{1}{4}[14 - 2(0) - 0]$$
$$= \frac{1}{4}[14 - 0 - 0]$$
$$= \frac{14}{4}$$
$$x^{(1)} = 3.50$$

$$y^{(1)} = \frac{1}{5}[10 - x^{(0)} + y^{(0)}]$$
$$= \frac{1}{5}[10 - 0 + 0]$$
$$= \frac{10}{5}$$
$$y^{(1)} = 2$$

$$z^{(1)} = \frac{1}{8}[20 - x^{(0)} - y^{(0)}]$$
$$= \frac{1}{8}[20 - 0 - 0]$$
$$= \frac{20}{8}$$
$$z^{(1)} = 2.5$$

2nd interation

$$x^{(2)} = \frac{1}{4}[14 - 2y^{(1)} - z^{(1)}]$$
$$= \frac{1}{4}[14 - 2(2) - 2.5]$$
$$= 1.875$$

$$y^{(2)} = \frac{1}{5}[10 - x^{(1)} + y^{(1)}]$$
$$= \frac{1}{5}[10 - 3.50 + 2]$$
$$= 1.70$$

$$z^{(2)} = \frac{1}{8}[20 - x^{(1)} - y^{(1)}]$$
$$= \frac{1}{8}[20 - 3.50 - 2]$$
$$= 1.8125$$

3rd interation

$$x^{(3)} = \frac{1}{4}[14 - 2y^{(2)} - z^{(2)}]$$
$$= \frac{1}{4}[14 - 2(1.70) - 1.8125]$$
$$= 2.1969$$

$$y^{(3)} = \frac{1}{5}[10 - x^{(2)} + y^{(2)}]$$
$$= \frac{1}{5}[10 - 1.875 + 1.70]$$
$$= 1.965$$

$z^{(3)} = \frac{1}{8}[20 - x^{(2)} - y^{(2)}]$

$z^{(3)} = 2.0531$

4th interation

$x^{(4)} = \frac{1}{4}[14 - 2y^{(3)} - z^{(3)}]$

$= \frac{1}{4}[14 - 2(1.965) - 2.0531]$

$= 2.0042$

$y^{(4)} = \frac{1}{5}[10 - x^{(3)} + z^{(3)}]$

$= \frac{1}{5}[10 - 2.1969 + 2.0531]$

$= 1.9712$

$z^{(4)} = \frac{1}{8}[20 - x^{(3)} - y^{(3)}]$

$= \frac{1}{8}[20 - 2.1969 - 1.965]$

$= 1.9798$

5th interiation

$x^{(5)} = \frac{1}{4}[14 - 2y^{(4)} - z^{(4)}]$

$x^{(5)} = \frac{1}{4}[14 - 2(1.9712) - 1.9798]$

$= 2.0195$

$y^{(5)} = \frac{1}{5}[10 - x^{(4)} + z^{(4)}]$

$= \frac{1}{5}[10 - 2.0042 + 1.9798]$

$= 1.9951$

$z^{(5)} = \frac{1}{8}[20 - x^{(4)} - y^{(4)}]$

$= \frac{1}{8}[20 - 2.0042 - 1.9712]$

$= 2.0031$

6th interation

$x^{(6)} = \frac{1}{4}[14 - 2y^{(5)} - z^{(5)}]$

$= \frac{1}{4}[14 - 2(1.9951) - 2.0031]$

$= 2.0017$

$y^{(6)} = \frac{1}{5}[10 - x^{(5)} + z^{(5)}]$

$= \frac{1}{5}[10 - \cancel{2.0195}\ 2.0195 + 2.0031]$

$= \cancel{1.95924} = 1.9967$

$z^{(6)} = \frac{1}{8}[20 - x^{(5)} - y^{(5)}]$

$= \frac{1}{8}[20 - 2.2069 - 1.9951]$

$= 1.9982$

7th interation

$x^{(7)} = \frac{1}{4}[14 - 2y^{(6)} - z^{(6)}]$

$= \frac{1}{4}[14 - 2(1.9967) - 1.9982]$

$= 2.0021$

$y^{(7)} = \frac{1}{5}[10 - x^{(6)} + z^{(6)}]$

$= \frac{1}{5}[10 - 2.0017 + 1.9982]$

$= 1.9993$

$z^{(7)} = \frac{1}{8}[20 - x^{(6)} - y^{(6)}]$

$= \frac{1}{8}[20 - 2.0017 - 1.9967]$

$= 2.0002$

8th interation

$x^{(8)} = \frac{1}{4}[14 - 2y^{(7)} - z^{(7)}]$

$= \frac{1}{4}[14 - 2(1.9993) - (2.0002)]$

$= 2.0003$

$y^{(8)} = \frac{1}{5}[10 - x^{(7)} + z^{(7)}]$

$= \frac{1}{5}[10 - 2.0021 + 2.0002]$

$= 1.9996$

$z^{(8)} = \frac{1}{8}[20 - x^{(7)} - y^{(7)}]$

$= \frac{1}{8}[20 - 2.0021 - 1.9993]$

$= 1.9998$

9th interation

$$x^{(9)} = \frac{1}{4}\left[14 - 2y^{(8)} - z^{(8)}\right]$$

$$= \frac{1}{4}\left[14 - 2(1.9996) - 1.9998\right]$$

$$= 2.0003$$

$$y^{(9)} = \frac{1}{5}\left[10 - x^{(8)} + z^{(8)}\right]$$

$$= \frac{1}{5}\left[10 - 2.0003 + 1.9998\right]$$

$$= 1.9999$$

$$z^{(9)} = \frac{1}{8}\left[20 - x^{(8)} - y^{(8)}\right]$$

$$= \frac{1}{8}\left[20 - 2.0003 - 1.9996\right]$$

$$= 2.0000$$

Iteration	x	y	z
1	3.50	2	1.5
2	1.875	1.70	1.8175
3	2.1969	1.965	2.0531
4	2.0042	1.9712	1.9798
5	2.0175	1.9951	2.0051
6	2.0017	1.9967	1.9982
7	2.0021	1.9993	2.0002
8	2.0003	1.9996	1.9998
9	2.003	1.9999	2.0000
10	2.0000	1.9999	1.9999

10th interation

$$x^{(10)} = \frac{1}{4}\left[14 - 2y^{(9)} - z^{(9)}\right]$$

$$= \frac{1}{4}\left[14 - 2[1.9999] - 2.0000\right]$$

$$= 2.0000$$

$$y^{(10)} = \frac{1}{5}\left[10 - x^{(9)} + z^{(9)}\right]$$

$$= \frac{1}{5}\left[10 - 2.0003 + 2.0000\right]$$

$$= 1.9999$$

$$z^{(10)} = \frac{1}{8}\left[20 - x^{(9)} - y^{(9)}\right]$$

$$= \frac{1}{8}\left[20 - 2.0003 - 2.0000\right]$$

$$= 1.9999$$

Solve the equations $20x+y-2z=17$ by gauss Jacobi iteration method.

$3x+20y-z=-18$

$2x-3y+20z=25$

sol Given equations

$$\left.\begin{aligned} 20x+y-2z&=17\\ 3x+20y-z&=-18\\ 2x-3y+20z&=25 \end{aligned}\right\} \rightarrow ①$$

eq ① can be written in the form

$$x=\frac{1}{20}[17-y+2z]$$

$$y=\frac{1}{20}[-18-3x+z]$$

$$z=\frac{1}{20}[25-2x+3y]$$

1st interation

we consider initial values for x, y & z is $x^0=0, y^0=0$ & $z^0=0$

$$x^{(1)}=\frac{1}{20}[17-y^{(0)}+2(z)^0]$$

$$=\frac{1}{20}[17-0+2(0)]$$

$$x'=0.85$$

$$y^{(1)}=\frac{1}{20}[-18-3x^{(0)}+z^{(0)}]$$

$$=\frac{1}{20}[-18-0+0]$$

$$=-0.9$$

$$z^{(1)}=\frac{1}{20}(25-2x^{(0)}+3y^{(0)}]$$

$$=\frac{1}{20}[25-0+0]$$

$$z^{(1)}=1.25$$

2nd interation

$$x^{(2)}=\frac{1}{20}[17-y^{(1)}+2z^{(1)}]$$

$$=\frac{1}{20}[17+0.9+2(1.25)]$$

$$=1.02$$

$$y^2=\frac{1}{20}[-18-3x^{(1)}+z^{(1)}]$$

$$=\frac{1}{20}[-18-3(0.85)+1.25]$$

$= -0.965$

$z^2 = \frac{1}{20}[25 - 2(x)^{(1)} + 3y^{(1)}]$

$= \frac{1}{20}(25 - 2(0.85) + 3(-0.9))$

$z^2 = 1.03$

3rd iteration

$x^3 = \frac{1}{20}(17 - y^{(2)} + 2z^{(2)})$

$= \frac{1}{20}(17 + 0.965 + 2(1.03))$

$= 1.001$

$y^3 = \frac{1}{20}[-18 - 3x^{(2)} + z^{(2)}]$

$= \frac{1}{20}[-18 - 3(1.02) + 1.03]$

$= -1.0015$

$z^3 = \frac{1}{20}[25 - 2x^{(2)} + 3y^{(2)}]$

$= \frac{1}{20}[25 - 2(1.02) + 3(-0.965)]$

$= 1.0032$

4th iteration

$x^4 = \frac{1}{20}[17 - y^{(3)} + 2z^{(3)}]$

$= \frac{1}{20}[17 + 1.0015 + 2(1.0032)]$

$= 1.0004$

$y^4 = \frac{1}{20}[-18 - 3x^{(3)} + z^{(3)}]$

$= \frac{1}{20}[-18 - 3(1.001) + 1.0032$

$= -0.9999$

$z^4 = \frac{1}{20}[25 (-2x^3 + 3y^{(3)}]$

$= \frac{1}{20}[25 - 2(1.001) + 3(-1.0015)]$

$= 0.9997$

5th iteration

$x^5 = \frac{1}{20}[17 - y^4 + 2z^{(4)}]$

$= \frac{1}{20}[17 + 0.9999 + 2(0.9997)]$

$= 0.9999$

$y^5 = \frac{1}{20}[-18 - 3x^{(4)} + z^{(4)}]$

$= \frac{1}{20}[-18 - 3(1.0004) + 0.9997)$

$= -1.0000$

$z^5 = \frac{1}{20}[25 - 2x^{(4)} + 3y^{(4)}]$

$= \frac{1}{20}[25 - 2(1.0004) + 3(0.9999)]$

$= 0.9999$

6th iteration

$x^{(6)} = \frac{1}{20}(17 - y^{(5)} + 2z^{(5)}]$

$= \frac{1}{20}[17 + 1.0000 + 2(0.9999)]$

$= 0.9999$

$y^{(6)} = \frac{1}{20}(-18 - 3x^{(5)} + z^{(5)}]$

$= \frac{1}{20}[-18 - 3(0.9999) + 0.9999]$

$= -0.9999$

$z^{(6)} = \frac{1}{20}[25 - 2x^{(5)} + 3y^{(5)}]$

$= \frac{1}{20}(25 - 2(0.9999) + 3(-1.0000))$

$= 1.0000$

7th iteration

$x^7 = \frac{1}{20}[17 - y^{(6)} + 2z^{(6)}]$

$= \frac{1}{20}[17 + 0.9999 + 2(1.0000)]$

$= 0.9999$

$y^7 = \frac{1}{20}[-18 - 3x^{(6)} + z^{(6)}]$

$= \frac{1}{20}[-18 - 3(0.9999) + 1]$

$= -0.9999$

$z^7 = \frac{1}{20}[25 - 2x^{(6)} + 3y^{(6)}]$

$= \frac{1}{20}[25 - 2(0.9999) + 3(-0.9999)]$

$= 1.0000$

iteration	x	y	z
1	0.85	-0.9	1.25
2	1.02	-0.965	1.03
3	1.001	-1.0015	1.0032
4	1.0004	-0.9999	0.9997
5	0.9999	-1.0000	0.9999
6	0.9999	-0.9999	1.0000
7	0.9999	-0.9999	1.0000

The required solution is

$x = 0.9999$, $y = -0.9999$

and $z = 1$

P

gaus 4) $10x + 5y -$

$4x - 10y$

$x + 6y$

gaus 5) $x_1 + 10x_2 +$

$10x_1 + x_2$

$x_1 + x_2 + 1$

Gauss siedel iterative method:

Let us consider a system of linear equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \rightarrow ①$$

eq ① can be written in the form

$$x = \frac{1}{a_{11}}[b_1 - a_{12}y - a_{13}z]$$

$$y = \frac{1}{a_{22}}[b_2 - a_{21}x - a_{23}z]$$

$$z = \frac{1}{a_{33}}[b_3 - a_{31}x - a_{32}y]$$

In the 1st iteration for $x^{(1)}$ we consider as $y^{(0)} = 0$ and $z^{(0)} = 0$. To determine $y^{(1)}$ value ~~to~~ we using $x^{(1)}$ & $z^{(0)}$ values. To determine $z^{(1)}$ value, we using $x^{(1)}$ & $z^{(1)}$

To continue this process until the difference of two consequtive approximation are coinside

Problems

1. solve the following equations by using gauss siedel iterative method.

$$4x+2y+z=14$$
$$x+5y-z=10$$
$$x+y+8z=20$$

Sol Given equations

$$\left.\begin{aligned} 4x+2y+z &= 14 \\ x+5y-z &= 10 \\ x+y+8z &= 20 \end{aligned}\right\} \rightarrow ①$$

eq ① can be written in the form

$$x = \frac{1}{4}[14 - 2y - z]$$

$$y = \frac{1}{5}[10 - x + z]$$

$$z = \frac{1}{8}[20 - x - y]$$

1st iteration

$$x^{(1)} = \frac{1}{4}(14 - 2y^{(0)} - z^{(0)})$$

$$= \frac{1}{4}[14 - 2(0) - (0)]$$

$$x^{(1)} = 3.5$$

$$y^{(1)} = \frac{1}{5}[10 - x^{(1)} + z^{(0)}]$$

$$= \frac{1}{5}[10 - 3.5 + 0]$$

$$y^{(1)} = 1.3$$

$$z^{(1)} = \frac{1}{8}[20 - x^{(1)} - y^{(1)}]$$

$$= \frac{1}{8}[20 - 3.5 - 1.3]$$

$$z^{(1)} = 1.9$$

2nd interation

$$x^{(2)} = \frac{1}{4}[14 - 2y^{(1)} - z^{(1)}]$$

$$= \frac{1}{4}(14 - 2(1.3) - 1.9]$$

$$x^{(2)} = 2.375$$

$$y^{(2)} = \frac{1}{5}[10 - x^{(2)} + z^{(1)}]$$

$$= \frac{1}{5}[10 - 2.375 + 1.9]$$

$$y^{(2)} = 1.905$$

$$z^{(2)} = \frac{1}{8}[20 - x^{(2)} - y^{(2)}]$$

$$= \frac{1}{8}[20 - 2.375 - 1.905]$$

$$z^{(2)} = 1.965$$

3rd iteration

$$x^{(3)} = \frac{1}{4}[14 - 2y^{(2)} - z^{(2)}]$$

$$= \frac{1}{4}[14 - 2(1.905) - 1.965]$$

$$x^{(3)} = 2.0562$$

$$y^{(3)} = \frac{1}{5}[10 - x^{(3)} + z^{(2)}]$$

$$= \frac{1}{5}[10 - 2.0562 + 1.965]$$

$= 1.9817$

$z^{(3)} = \frac{1}{8}\left[20 - x^{(3)} - y^{(3)}\right]$

$= \frac{1}{8}\left[20 - 2.0562 - 1.9817\right]$

$= 1.9953$

4^{th} iteration

$x^{(4)} = \frac{1}{4}\left[14 - 2y^{(3)} - y^{(3)}\right]$

$= \frac{1}{4}\left[14 - 2(1.9817) - 1.9953\right]$

$x^{(4)} = 2.0103 \simeq 2$

$y^{(4)} = \frac{1}{5}\left[10 - x^{(4)} + z^{(3)}\right]$

$= \frac{1}{5}\left[10 - 2.0103 + 1.9953\right]$

$y^{(4)} = 1.997 \simeq 2$

$z^{(4)} = \frac{1}{8}\left[20 - x^{(4)} - y^{(4)}\right]$

$= \frac{1}{8}\left[20 - 2.0103 - 1.997\right]$

$z^{(4)} = 1.9993 \simeq 2$

5^{th} iteration

$x^{(5)} = \frac{1}{4}\left[14 - 2y^{(4)} - z^{(4)}\right]$

$= \frac{1}{4}\left[14 - 2(1.997) - 1.9993\right]$

$= 2.0017 \simeq 2$

$y^{(5)} = \frac{1}{5}\left[10 - x^{(5)} + y^{(4)}\right]$

$= \frac{1}{5}\left[10 - 2.0017 + 1.997\right]$

$= 1.9991 \simeq 2$

$z^{(5)} = \frac{1}{8}\left[20 - x^{(5)} - y^{(5)}\right]$

$= \frac{1}{8}\left[20 - 2.0017 - 1.9991\right]$

$z^{(5)} = 1.9999 \simeq 2$

iteration	x	y	z
1	3.5	1.3	1.9
2	2.375	1.905	1.965
3	2.0562	1.9817	1.9953
4	2.0103	1.997	1.9993
5	2.0017	1.9991	1.9999

$x = 2.0017 \quad y = 1.9991 \quad z = 1.9999$

3. solve the following equations by using gauss siedel iterative method $27x+6y-z=85$

$$x+y+54z=110$$

$$6x+15y+2z=72$$

Given equations

$$27x+6y-z=85$$
$$x+y+54z=110$$
$$6x+15y+2z=72$$

By backward direction.

$$27x+6y-z=85$$
$$6x+15y+2z=72$$
$$x+y+54z=110$$

eq ① we can be written in the form

$$x=\frac{1}{27}[85-6y+z]$$

$$y=\frac{1}{15}[72-6x-2z]$$

$$z=\frac{1}{54}[110-x-y]$$

1st iteration

$$x^{(1)}=\frac{1}{27}[85-6y^{(0)}-z^{(0)}]$$

$$=\frac{1}{27}[85-0-0]$$

$$=3.1481$$

$$y^{(1)}=\frac{1}{15}[72-6(x)^{1}-2(z)^{0}]$$

$$=\frac{1}{15}[72-6-0]$$

$$=4.4$$

$$z^{(1)}=\frac{1}{54}[110-x^{(1)}-y^{(1)}]$$

$$=\frac{1}{54}(110-(3.1481)-(4.4))$$

$$=1.8974$$

2nd iteration

$$x^{(2)}=\frac{1}{27}[85-6y^{(1)}-z^{(1)}]$$

$$=\frac{1}{27}[85-6(4.4)-(1.8974))$$

$$=2.1000$$

$$y^{(2)}=\frac{1}{15}(72-6(x)^{2}-2(z)^{1})$$

$$=\frac{1}{15}(72-6(2.1000)-2(1.8974))$$

$$=3.7070$$

$$z^{(2)}=\frac{1}{54}[110-x^{(2)}y^{(2)}]$$

$$=\frac{1}{54}[110-2.1000-3.7070]$$

$$=1.9295$$

3rd iteration

$$x^{(3)}=\frac{1}{27}[85-6y^{(2)}-z^{(2)}]$$

$$=\frac{1}{27}(85-6(3.7070)-(1.9295))$$

$$=2.2529$$

$$y^{(3)}=\frac{1}{15}(72-6(x)^{3}-2(z)^{2})$$

$$=\frac{1}{15}(72-6(2.2529)-2(1.9295))$$

$$=3.6416$$

$$z^{(3)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right]$$
$$= \frac{1}{54}\left[110 - (2.2529) - (3.6416)\right]$$
$$= 1.9279$$

4th iteration

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} - z^{(3)}\right]$$
$$= \frac{1}{27}\left[85 - 6(3.6416) - (1.9279)\right]$$
$$= 2.2675$$
$$y^{(4)} = \frac{1}{15}\left[72 - 6(x)^{4} - 2(z)^{3}\right]$$
$$= \frac{1}{15}\left[72 - 6(2.2675) - 2(1.9279)\right]$$
$$= 3.6359$$
$$z^{(4)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right]$$
$$= \frac{1}{54}\left[110 - 2.2675 - 3.6359\right]$$
$$= 1.9277$$

5th iteration

$$x^{(5)} = \frac{1}{27}\left[85 - 6y^{(4)} - z^{(4)}\right]$$
$$= \frac{1}{27}\left[85 - 6(3.6359) - (1.9277)\right]$$
$$= 2.2688$$
$$y^{(5)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(4)}\right]$$
$$= \frac{1}{15}\left[72 - 6(2.2688) - 2(1.9277)\right]$$
$$= 3.6354$$
$$z^{(5)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right]$$
$$= \frac{1}{54}\left[110 - (2.2688) - (3.6354)\right]$$
$$= 1.9277$$

iteration	x	y	z
1	3.1481	4.4	1.8974
2	2.1000	3.7070	1.9295
3	2.2529	3.6416	1.9274
4	2.2675	3.6359	1.9277
5	2.2688	3.6354	1.9277

$x = 2.2688 \quad y = 3.6354 \quad z = 1.9277$

4. $x_1 + 10x_2 + x_3 = 6$
$10x_1 + x_2 + x_3 = 6$
$x_1 + x_2 + 10x_3 = 6$

Sol) Given equations

$x_1 + 10x_2 + x_3 = 6$
$10x_1 + x_2 + x_3 = 6$
$x_1 + x_2 + 10x_3 = 6$

By Backward direction

$$\left.\begin{array}{l} 10x_1 + x_2 + x_3 = 6 \\ x_1 + 10x_2 + x_3 = 6 \\ x_1 + x_2 + 10x_3 = 6 \end{array}\right\} \to (1)$$

eq (1) can be written in the form

$$x_1 = \frac{1}{10}\left[6 - x_2 - x_3\right]$$
$$x_2 = \frac{1}{10}\left[6 - x_1 - x_3\right]$$
$$x_3 = \frac{1}{10}\left[6 - x_1 - x_2\right]$$

1st iteration

$$x_1^{(1)} = \frac{1}{10}\left[6 - x_2^{(0)} - x_3^{(0)}\right]$$
$$= \frac{1}{10}\left[6 - 0 - 0\right]$$
$$= 0.6$$
$$x_2^{(1)} = \frac{1}{10}\left[6 - x_1^{(1)} - x_3^{(0)}\right]$$
$$= \frac{1}{10}\left[6 - 0.6 - 0\right]$$
$$= 0.54$$
$$x_3^{(1)} = \frac{1}{10}\left[6 - x_1 - x_2\right]$$
$$= \frac{1}{10}\left[6 - 0.6 - 0.54\right]$$
$$= 0.486$$

2nd iteration

$$x_1^{2} = \frac{1}{10}\left[6 - x_2^{(1)} - x_3^{(1)}\right]$$
$$= \frac{1}{10}\left[6 - 0.54 - 0.486\right]$$
$$= 0.4974$$
$$x_2^{(2)} = \frac{1}{10}\left[6 - x_1^{(2)} - x_3^{(1)}\right]$$
$$= \frac{1}{10}\left(6 - 0.4974 - 0.486\right)$$
$$= 0.5016$$
$$x_3^{(2)} = \frac{1}{10}\left(6 - x_1^{2} - x_2^{2}\right)$$
$$= \frac{1}{10}\left(6 - 0.497 - 0.5016\right]$$
$$= 0.5001$$

3rd iteration

$$x_1^{(3)} = \frac{1}{10}\left[6 - x_2^{2} - x_3^{(2)}\right]$$
$$= \frac{1}{10}\left(6 - 0.5016 - 0.5001\right]$$

$= 0.4999$

$x_2^{(3)} = \frac{1}{10}(6 - x_1^3 - x_3^2)$

$= \frac{1}{10}(6 - 0.4999 - 0.5001)$

$= 0.5$

$x_3^{(3)} = \frac{1}{10}(6 - x_1^3 - x_2^3)$

$= \frac{1}{10}[6 - 0.4999 - 0.5]$

$= 0.5000$

4th iteration

$x_1^4 = \frac{1}{10}[6 - x_2^3 - x_3^3]$

$= \frac{1}{10}[6 - 0.5 - 0.5000]$

$= 0.5$

$x_2^4 = \frac{1}{10}[6 - x_1^4 - x_3^3]$

$= \frac{1}{10}[6 - 0.5 - 0.5]$

$= 0.5$

$x_3^4 = \frac{1}{10}[6 - x_1^4 - x_2^4]$

$= \frac{1}{10}(6 - 0.5 - 0.5)$

$= 0.5$

5th iteration

$x_1^5 = \frac{1}{10}[6 - x_2^4 - x_3^4]$

$= \frac{1}{10}[6 - 0.5 - 0.5]$

$= 0.5$

$x_2^5 = \frac{1}{10}(6 - x_1^5 - x_3^4)$

$= \frac{1}{10}[6 - 0.5 - 0.5]$

$= 0.5$

$x_3^5 = \frac{1}{10}(6 - x_1^5 - x_2^5)$

$= \frac{1}{10}(6 - 0.5 - 0.5)$

$= 0.5$

Iteration	x	y	z
1	0.6	0.54	0.486
2	0.4974	0.5016	0.5001
3	0.4999	0.5	0.5
4	0.5	0.5	0.5
5	0.5	0.5	0.5

2. Eigen values, Eigen vectors & Orthogonal transformation

Characteristic equation:- The equation $|A-\lambda I|=0$ is called characteristic equation of a matrix A.

Eigen values:- The roots of characterstic equation is called an A. Eigen values (or) characterstic roots (or) latent roots of a matric "A".

Eigen vector:- Let "A" is a $n \times n$ matrix, a non-zero vector x is said to be characteristic vector or Eigen vector of "A".

If λ is a scalar (Eigen value) such that $AX = \lambda x$ $(x \neq 0)$.

If $AX = \lambda X$, we say that x is eigen vector of "A". corresponding to the eigen value.

Properties of Eigen values & Eigen vectors

1) Sum of Eigen values of "A" is equal to the trace of a matrix "A" (trace is nothing but) sum of diagonals elements of "A".

2) The product of Eigen values of a matrix "A" is equal to the determinent of matrix "A".

3) Eigen values of "A" is same as Eigen values of A^T.

4) If λ is a an Eigen value of "A" then λ^{-1} (inverse) is an Eigen value of A^{-1} and corresponding vector of A & A^{-1} is x itself.

5) If λ is an Eigen value of "A" then the Eigen value of B is $B = a_0 A^2 + a_1 A + a_2 I$ is $a_0\lambda^2 + a_1\lambda + a_2$

B is inverse of A.

Problems

find the Eigen value of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Given matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

w.k.t The characteristic eqn of A

is $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \overset{+}{1-\lambda} & \overset{-}{2} & \overset{+}{-3} \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} 3-\lambda & 2 \\ 0 & -2-\lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ 0 & -2-\lambda \end{vmatrix} - 3 \begin{vmatrix} 0 & 3-\lambda \\ 0 & 0 \end{vmatrix} = 0$$

$$(1-\lambda) \cdot [(3-\lambda)(-2-\lambda) - 0] - 2[0-0] - 3[0-0] = 0$$

$$\Rightarrow (1-\lambda) [(3-\lambda)(-2-\lambda)] = 0$$

$$\Rightarrow (1-\lambda) [-6 - 3\lambda + 2\lambda + \lambda^2] = 0$$

$$\Rightarrow (1-\lambda) (\lambda^2 + 2\lambda - 3\lambda - 6] = 0$$

$$\Rightarrow (1-\lambda) [\lambda(\lambda+2) - 3(\lambda+2)] = 0$$

$$\Rightarrow (1-\lambda) (\lambda+2) (\lambda-3)] = 0$$

$$(1-\lambda) (\lambda+2) (\lambda-3) = 0$$

$$\Rightarrow (1-\lambda) = 0 \Rightarrow \lambda = 1$$

$\Rightarrow (\lambda+2)=0 \Rightarrow \lambda=-2$

$\Rightarrow (\lambda+3)=0 \Rightarrow \lambda=3$

$\therefore$ The eigen values of A is

$\lambda = 1, -2, 3$

2. $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

w.k.That the characteristic eqn of A

is $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\left| \begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow (-1-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 0 & -\lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & -1-\lambda \\ 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda) \left| (-1-\lambda)(-\lambda) - 0 \right] - 1 [\lambda - 0] + 0 [0-0] = 0$$

$$\Rightarrow (-1-\lambda) [(\lambda+\lambda^2)] - 1 [8-\lambda] = 0$$

$$-1-\lambda[\lambda+\lambda^2]+\lambda=0$$
$$-\lambda-\lambda^2-\lambda^2-\lambda^3+\lambda=0$$
$$-2\lambda^2-\lambda^3=0$$
$$-\lambda[2\lambda+\lambda^2]=0$$
$$\boxed{\lambda=0} \quad \lambda(2+\lambda)=0$$
$$\boxed{\lambda=0} \quad 2+\lambda=0 \quad \boxed{\lambda=-2}$$

Find the eigen values and eigen vectors of the matrix (compulsory) $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$

W.K.T.

The characteristic eqn of A is $|A-\lambda I|=0$

$$\Rightarrow \left|\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right|=0$$

$$\Rightarrow \left|\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right|=0$$

$$\Rightarrow \begin{vmatrix}\overset{+}{8-\lambda} & \overset{-}{-6} & \overset{+}{2}\\ -6 & 7-\lambda & -4\\ 2 & -4 & 3-\lambda\end{vmatrix}=0$$

$$\Rightarrow (8-\lambda)\begin{vmatrix}7-\lambda & -4\\ -4 & 3-\lambda\end{vmatrix}+6\begin{vmatrix}-6 & -4\\ 2 & 3-\lambda\end{vmatrix}+2\begin{vmatrix}-6 & 7-\lambda\\ 2 & -4\end{vmatrix}=0$$

$$\Rightarrow (8-\lambda)|(7-\lambda)(3-\lambda)-16|+6(-6(3-\lambda)+8|+2|24-2(7-\lambda)|=0$$

$$\Rightarrow (8-\lambda)|(21-7\lambda-3\lambda+\lambda^2)-16|+6[(-18+6\lambda)+8|+2|24-14+2\lambda|=0$$

$$\Rightarrow (8-\lambda)[\lambda^2-10\lambda+5]+6[6\lambda-10]+2[2\lambda+10]=0$$

$$\Rightarrow 8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0$$

$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$

$\Rightarrow -\lambda(\lambda^2-18\lambda+45)=0$

$\Rightarrow -\lambda(\lambda^2-3\lambda-15\lambda+45)=0$

$\Rightarrow -\lambda[\lambda(\lambda-3)-15(\lambda-3)]=0$

$\Rightarrow -\lambda[(\lambda-3)(\lambda-15)]=0$

$\Rightarrow -\lambda=0 \Rightarrow \lambda=0$

$\Rightarrow \lambda-3=0 \Rightarrow \lambda=3$

$\Rightarrow \lambda-15=0 \Rightarrow \lambda=15$

$\therefore \lambda = 0,3,15$

Hence, the eigen values are $0,3,15$ //.

Eigen vector

Sol Given matrix : To find eigen vector of the given matrix "A".

$A = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

Case i: Eigen vector corresponding to $\lambda=0$.

The equation is $[A-\lambda I]X=0$.

Given matrix

$$A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Eigen vector of A is

$[A-\lambda I]X=0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \rightarrow (1)$$

sub $\lambda=0$ in (1)

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$8x - 6y + 2z = 0 \rightarrow (2)$

$-6x + 7y - 4z = 0 \rightarrow (3)$

$2x - 4y + 3z = 0 \rightarrow (4)$

from (2) & (3)

$$\begin{array}{cccc} x & y & z & \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{array}$$

$$\frac{x}{24-14} = \frac{y}{-12+32} = \frac{z}{56-36}$$

$$\Rightarrow \frac{x}{\cancel{10}_1} = \frac{y}{\cancel{20}_2} = \frac{z}{\cancel{20}_2}$$

$x = 1, \ y = 2, \ z = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

case ii) :- Eigen vector corresponding to $\lambda = 3$

The equation is $[A - \lambda I]\, x = 3$

sub λ in (1)

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow \ 5x - 6y + 2z = 0 \rightarrow (2)$

$-6x + 4y - 4z = 0 \rightarrow (3)$

$2x - 4y + 0z = 0 \rightarrow (4)$

solving ② & ③

$$\begin{array}{cccc} x & y & z & \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 41 \end{array}$$

$$\frac{x}{24-8} = \frac{y}{-12+20} = \frac{z}{20-36}$$

$$\frac{x}{\cancel{16}_{2}} = \frac{y}{\cancel{8}_{1}} = \frac{z}{-\cancel{16}_{2}}$$

$x = 2, \quad y = 1, \quad z = -2$

$x = 2, \ y = 1, \ z = -2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

case iii) Eigen vector corresponding to $\lambda = 15$

The equation is $[A - \lambda I]X = 15$

sub $\lambda = 15$ in ①.

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-7x - 6y + 2z = 0 \rightarrow (2)$

$-6x - 8y - 4z = 0 \rightarrow (3)$

$2x - 4y - 12z = 0 \rightarrow (4)$

solving (3) & (4)

$$\begin{array}{cccc} x & y & z & \\ -8 & -4 & -6 & -8 \\ -4 & -12 & 2 & -4 \end{array}$$

$$\frac{x}{96-16} = \frac{y}{-8-72} = \frac{z}{24+16}$$

$$\Rightarrow \frac{x}{\cancel{80}_{2}} = \frac{y}{\cancel{-80}_{2}} = \frac{z}{\cancel{40}_{1}}$$

$$x = 2, \quad y = -2, \quad z = 1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

5. find the eigen values values and the corresponding eigens vectors of $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

sol

Given $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristic eqn of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - \lambda - 6\lambda + 6 = 0$$

$$\lambda(\lambda - 1) - 6(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\therefore \lambda = 1, 6$$

Hence, the eigen values of A one 1, 6

To find eigen vectors

Eigen vector corresponding to $\lambda = 1$

$$\Rightarrow (A - \lambda I)x = 0$$

$$\Rightarrow (A - 1 \cdot I)x = 0$$

$$\left[\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The system of eqns is

$$4x + 4y = 0$$

$$x + y = 0$$

$$x = -y$$

Taking $x = k$

$$k = -y$$

$$\boxed{y = -k}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} k$$

Eigen vector corresponding to $\lambda = 6$

$$(A - 6I) X = 0$$

$$\left[\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x + 4y = 0 \Rightarrow -x = -4y$$

$$x - 4y = 0 \qquad x = 4y$$

Taking $y = k \Rightarrow x = 4k$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} k$$

Hence $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigen vector of A corresponding the eigen value $\lambda = 6$.

6. find the eigen values and corresponding eigen vectors of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

sol Given $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic eq is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)\left[(1-\lambda)(1-\lambda) - 12\right] - 2\left[-2\lambda - 6\right] - 3\left[-4 + 1\cancel{\lambda}\right]$$

$$4\lambda + 12 \qquad 4(\lambda + 3)$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]+4(\lambda+3)+3(\lambda+3)=0$$

$$\Rightarrow -(\lambda+2)(\lambda-4)(\lambda+3)+7(\lambda+3)=0$$

$$\Rightarrow (\lambda+3)\left[-(\lambda+2)(\lambda-4)+7\right]=0$$

$$\Rightarrow -(\lambda+3)\left(\lambda^2-2\lambda-15\right]=0$$

-15

-5 +3

$$(\lambda+3)(\lambda+3)(\lambda-5)=0$$

$$\boxed{\lambda=-3,-3,5}$$

To find eigen vectors

If x is an eigen vector of A corresponding to the eigen value λ of A, we have

$$(A-\lambda I)x=0$$

corresponding eigen vector for $\lambda=-3$

$$(A-(-3)I)x=0$$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Put $\lambda=-3$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{matrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{matrix}\right.\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$R_2 \rightarrow R_2-2R_1$$

$$R_3 \rightarrow R_3+R_1$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x+2y-3z=0$$

$\Rightarrow r=1$

$n=3$

$n-r=3-1=2$ arbitory constants

Let $y = k_1$ & $z = k_2$

$x + 2k_1 - 3k_2 = 0$

$x = -2k_1 + 3k_2$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k_1 + 3k_2 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} k_2$$

Eigen vector corresponding to $\lambda = 5$

$(A - 5I)X = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow R_1 \leftrightarrow R_3$

$$\Rightarrow \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 7R_1$

$R_2 \rightarrow R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}$$

$\frac{R_2}{-8}, \frac{R_3}{16}$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 0 & +1 & +2 \\ 0 & 1 & 2 \end{bmatrix}$$

$R_3 \rightarrow R - R_2$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank $r = 2$

$n = 3$

$n - r = 3 - 2 = 1$

consider, arbitary constant

Let $z = k$

$$x + 2y + 5z = 0$$

$$y + 2z = 0$$

$$y + 2k = 0$$

$$\boxed{y = -2k}$$

$$x + 2(-2k) + 5k = 0$$

$$x - 4k + 5k = 0$$

$$x + k = 0$$

$$\boxed{x = -k}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} k$$

7. find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Sol

Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic eqn $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$

$\Rightarrow (6-\lambda)[9 + \lambda^2 - 6\lambda - 1] + 2[-6 + 2\lambda + 2] + 2[2 - 6 + 2\lambda] = 0$

$\Rightarrow (6-\lambda)[\lambda^2 - 6\lambda + 8] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$

$\Rightarrow (6-\lambda)[\lambda^2 - 2\lambda - 4\lambda + 8] + 4\lambda - 8 + 4\lambda - 8 = 0$

$\Rightarrow (6-\lambda)[\lambda(\lambda-2) - 4(\lambda-2)] + 8\lambda - 16 = 0$

$\Rightarrow (6-\lambda)[(\lambda-2)(\lambda-4)] + 8(\lambda-2) = 0$

$\Rightarrow (\lambda-2)\,[(\lambda-2)(\lambda-4)] + 8(\lambda-2) = 0$

$\Rightarrow (\lambda-2)\,[(6-\lambda)(\lambda-4)+8] = 0$

$\Rightarrow (\lambda-2)\,[-\lambda^2+4\lambda+6\lambda-24+8] = 0$

$\Rightarrow (\lambda-2)\,[-\lambda^2+10\lambda-16] = 0$

$\Rightarrow -(\lambda-2)\,(\lambda^2-10\lambda+16] = 0$

$\Rightarrow -(\lambda-2)\,(\lambda-2)(\lambda-8) = 0$

-16 ; -8, -2

$\lambda-2=0 \Rightarrow \lambda=2$

$\lambda-2=0 \Rightarrow \lambda=2$

$\lambda-8=0 \Rightarrow \lambda=8$

$\therefore \boxed{\lambda = 2,2,8}$

The eigen values of A are 2,2,8

To find eigen vectors.

case(i): Corresponding eigen vector for $\lambda=2$

$(A-\lambda I)X=0$

Put $\lambda=2$

$$\left(\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\right)\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{bmatrix}4 & -2 & 2\\ -2 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$R_3 \to 2R_3 + R_1$

$$\begin{bmatrix}4 & -2 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$r=1, \quad n=3$

$n-r = 3-1 = 2$

Let $y=k_1, \; z=k_2$

$4x-2y+2z=0$

$4x-2k_1+2k_2=0$

$4x = 2k_1 - 2k_2$

$x = 1/2\, k_1 - 1/2\, k_2$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2\, k_1 - 1/2\, k_2 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} k/2$$

case (ii)

corresponding eigen vector for $\lambda = 8$

$(A - 8I)x = 0$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r = 2,\ n = 3 \Rightarrow n - r = 3 - 2 = 1$

Let $\boxed{z = k}$

$-2x - 2y + 2z = 0$

$-3y - 3z = 0$

$-3y - 3k = 0$

$-3y = 3k$

$\boxed{y = -k}$

$-2x - 2(-k) + 2k = 0$

$-2x + 2k + 2k = 0$

$-2x + 4k = 0$

$-2x = -4k$

$\boxed{x = 2k}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} k$$

10. find the eigen values (or) eigen roots (or) characteristic root and the corresponding eigen vectors of the matrix 'A' & S, $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Given that,

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

we know that the characteristic eqn for the given matrix 'A' is $|A - \lambda I| = 0$

Now,

$$\left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4-0 \\ 3-0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$-10 \times 1 = -10$

2 -5

$$\Rightarrow \lambda^2 + 2\lambda - 5\lambda - 10 = 0$$

$$\Rightarrow \lambda(\lambda+2) - 5(\lambda+2) = 0$$

$$\Rightarrow \lambda + 2 = 0 \quad \lambda - 5 = 0$$

$$\boxed{\lambda = 2} \quad \boxed{\lambda = 5}$$

$\therefore \lambda_1 = -2, \lambda_2 = 5$ are the eigen values for the given of matrix 'A'.

if $\lambda_1 = -2$

The eigen vector of the matrix x 'A' can be obtained from the eqn $(A - \lambda I)x = 0$

Now

$$\left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1+2 & 4-0 \\ 3-0 & 3+2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x+4y=0$$
$$3x+4y=0$$

Let $y=k_1$

$$3x+4(k)=0$$
$$3x=-4k$$
$$x=-4/3\,k$$

$$X=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -4/3\,k \\ 1 \end{bmatrix}=k\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$$

$$\therefore X=\begin{bmatrix} x \\ y \end{bmatrix}=k\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$$

if $\lambda_2=5$:-

The eigen vectors of the given matrix 'A' is to be obtained from eq $(A-\lambda_2 I)X=0$

Now, $$\left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}-\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-5 & 4-0 \\ 3-0 & 2-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x+4y=0$$
$$3x-3y=0$$

Let, $y=k$

$$3x-3y=0$$
$$3x-3(k)=0$$
$$3x=3k$$
$$\boxed{x=k}$$

$$\therefore X=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} k \\ k \end{bmatrix}$$

$$\Rightarrow \therefore X=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ //.

11. find the characteristics roots & eigen vectors of and $A\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$.

Sol: Given that

$$A=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$$

w.k.T. The characteristics eqn for the given matrix 'A' is $|A-\lambda I|=0$

Now,

$$\left|\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right|=0$$

$$\Rightarrow \begin{bmatrix}5-\lambda & 4\\ 1 & 2-\lambda\end{bmatrix}=0$$

$$\Rightarrow (5-\lambda)(2-\lambda)-4=0$$

$$\Rightarrow 10-5\lambda-2\lambda+\lambda^2-4=0$$

$$\Rightarrow 10-7\lambda+\lambda^2-4=0$$

$$\Rightarrow \lambda^2-7\lambda+\cancel{10}=0.$$

$$\Rightarrow \lambda^2-1\lambda-6\lambda+6=0$$

$$\Rightarrow \lambda(\lambda-1)-6(\lambda-1)=0$$

$$\Rightarrow (\lambda-1)(\lambda-6)=0$$

$$\boxed{\lambda=1} \quad \boxed{\lambda=6}$$

$\therefore \lambda_1=1$, & $\lambda=6$ are the eigen values for the given of matrix 'A'.

if $\lambda_1=1$

The eigen vectors of the matrix 'A' can be obtained from the eqn $(A-\lambda_1 I)x=0$,

$$\left(\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}5-1 & 4-0\\ 1-0 & 2-1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$4x + 4y = 0$

$x + y = 0.$

Let $y = k$, $\Rightarrow 4x + 4(k) = 0$

$4x = -4k$

$x = -4k/4$

$\boxed{x = -k}$

If $\lambda_2 = 6$

The eigen values of the given matrix 'A' is to be obtained from $(A - \lambda_2 I)X = 0$

Now, $\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 5-6 & 4-0 \\ 1-0 & 2-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-1x + 4y = 0$ & $x - 4y = 0.$

Let $y = k$,

$-x + 4k = 0$

$+x = +4k$

$\boxed{x = 4k}$ &

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4k \\ 1 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Cayley Hamilton theorem

Every square matrix satisfy its own characteristic equation. i.e if "A" is a square matrix of order n and its characteristic equation is $|A-\lambda I| = (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n] = 0$.

Here $a_1, a_2, \ldots a_n$ are as coffecients then the corresponding matrix equation is $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n = 0$.

Applications

* To find the inverse of a matrix.
* To find higher powers of the matrix

Problem

1) If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ verify cayley hamilton theorem and find A^{-1}.

Sol Given

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

The characteristic eqn of A is

$$|A - I\lambda| = 0$$

$$\left| \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(-2-\lambda) - 0\right] - 1\left[5(-2-\lambda) + 3\right] + 2\left[0 + 3 - \lambda\right] = 0$$

$$\Rightarrow (2-\lambda)\left[-6-3\lambda+2\lambda+\lambda^2\right]+10+5\lambda-3+6-2\lambda=0$$

$$\Rightarrow (2-\lambda)\left[\lambda^2-\lambda-6\right]-3\lambda+13=0$$

$$\Rightarrow 2\lambda^2-2\lambda-12-\lambda^3+\lambda^2+6\lambda-3\lambda+13=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+7\lambda+1=0$$

$$\Rightarrow \lambda^3-3\lambda^2-7\lambda-1=0$$

To verify cayley Hamiltan theorem

we have, $A^3-3A^2-7A-I=0 \longrightarrow ①$

$$A^2=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$$

$$A^2=\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$$

$$A^3=\begin{bmatrix}36&22&23\\101&64&60\\-7&-3&-7\end{bmatrix}$$

from ①

$$=\begin{bmatrix}36&22&23\\101&64&60\\-7&-3&-7\end{bmatrix}-3\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}-7\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}36-21-14-1&22-15-7-0&23-9-14-0\\101-66-35-0&64-42-21-1&60-39-21-0\\-7-0+7-0&-3+3-0-0&-7-6+14-1\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The given matrix A satisfy cayley Hamilton theorem.

To find A^{-1}

from ①

$$A^3 - 3A^2 - 7A - I = 0$$

multiply by A^{-1} on b.s $\qquad A^{-1} = \frac{1}{A}$

$$A^{-1}(A^3 - 3A^2 - 7A - I) = 0 \qquad \frac{1}{A}$$

$$A^2 - 3A - 7I - A^{-1} = 0$$

$$A^2 - 3A - 7I = A^{-1}$$

$$A^{-1} = A^2 - 3A - 7I$$

$$A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

2. find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ by using cayley Hamilton theorem.

Sol: Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

The characteristic eqn of A is

$$|A - I\lambda| = 0$$

$$\left| \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(4-\lambda)(6-\lambda)-25\right]-2\left[2(6-\lambda)-15\right]+3\left[10-3(4-\lambda)\right]$

$\Rightarrow (1-\lambda)\left[24-4\lambda-6\lambda+\lambda^2-25\right]-2\left[12-2\lambda-15\right]+3\left[10-12+3\lambda\right]$

$\Rightarrow (1-\lambda)\left[\lambda^2-10\lambda-1\right]-2\left[-2\lambda-3\right]+3\left[3\lambda-2\right]=0$

$\Rightarrow (1-\lambda)(\lambda^2-10\lambda-1)+4\lambda+6+9\lambda-6=0$

$\Rightarrow \lambda^2-10\lambda-1-\lambda^3+10\lambda^2+\lambda+4\lambda+6+9\lambda-6=0$

$\Rightarrow -\lambda^3+11\lambda^2+4\lambda+I=0$

To verify cayley Hamilton theorem, so we S.T

$A^3-11A^2-4A-I=0 \rightarrow (1)$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

from (1)

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 157-154-4+1 & 283-275-8+0 & 353-341-12+0 \\ 283-275-8+0 & 510-495-16+1 & 636-616-20+0 \\ 353-34-12+0 & 636-616-20+0 & 793-770-24+1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, cayley hamilton theorem verified

To find A^{-1}

The characteristic eqn

$A^3 - 11A^2 + 4A + I = 0$

multiply by A^{-1}

$A^{-1}(A^3 - 11A^2 - 4A + I) = 0$

$A^2 - 11A - 4I + A^{-1} = 0$

$A^{-1} = -A^2 + 11A + 4I$

$$A^{-1} = -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

3. find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 5 \end{bmatrix}$ by using cayley - Hamilton theorem.

Sol: Given $A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 5 \end{bmatrix}$

The characteristic eqn of A is

$|A - \lambda I| = 0$

$$\left|\begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & -1 \\ 3 & 4-\lambda & 5 \\ -1 & 5 & 2-\lambda \end{vmatrix}$$

$\Rightarrow 1-\lambda\left[(4-\lambda)(2-\lambda)-20\right]-3\left[3(2-\lambda)+5\right]-1\left[15+1(4-\lambda)\right]$

$\Rightarrow 1-\lambda\left[8-4\lambda-2\lambda+\lambda^2-25\right]-3\left[6-3\lambda+5\right]-1\left[15+4-\lambda\right]$

$\Rightarrow (1-\lambda)\left[\lambda^2-6\lambda-17\right]-3\left[-3\lambda+11\right]-1\left[-\lambda+19\right]$

$\Rightarrow (1-\lambda)\left[\lambda^2-6\lambda-17\right]+9\lambda-33+\lambda-19$

$\Rightarrow \lambda^2-6\lambda-17-\lambda^3+6\lambda^2+17\lambda+9\lambda-33+\lambda-19$

$\Rightarrow -\lambda^3+7\lambda^2+21\lambda-69$

To verify cayley hamilton theorem

so, we show that

$$A^3-7A^2-21A+69I=0 \rightarrow (1)$$

Now,

$$A^2 = A\cdot A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 10 & 12 \\ 10 & 50 & 27 \\ 12 & 27 & 30 \end{bmatrix}$$

$$A^3 = A^2\cdot A = \begin{bmatrix} 11 & 10 & 12 \\ 10 & 50 & 27 \\ 12 & 27 & 30 \end{bmatrix}\begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 133 & 63 \\ 133 & 365 & 294 \\ 63 & 294 & 183 \end{bmatrix}$$

from (1)

$$= \begin{bmatrix} 29 & 133 & 63 \\ 133 & 365 & 294 \\ 63 & 294 & 183 \end{bmatrix} - 7\begin{bmatrix} 11 & 10 & 12 \\ 10 & 50 & 27 \\ 12 & 27 & 30 \end{bmatrix} - 21\begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 2 \end{bmatrix} + 69\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29-77-21+69 & 133-70-63+0 & 63-84+21+0 \\ 133-70-63+0 & 365-350-84+69 & 294-189-105+0 \\ 63-84+21+0 & 294-189-105+0 & 183-210-42+69 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Given matrix satisfies its characteristic equation

cayley hamilton theorem is verified

To find A^{-1}

we have $A^3 - 7A^2 - 21A + 69I = 0$

multiply by A^{-1}

$$A^{-1}(A^3 - 7A^2 - 21A + 69I) = 0$$

$$A^2 - 7A - 21I + 69A^{-1} = 0$$

$$69A^{-1} = -A^2 + 7A + 21I$$

$$A^{-1} = \frac{1}{69}\left[-A^2 + 7A + 21I\right]$$

$$\therefore A^{-1} = \frac{1}{69}\begin{bmatrix} -11 & -10 & -12 \\ -10 & -50 & -27 \\ -12 & -27 & -30 \end{bmatrix} + \begin{bmatrix} 7 & 21 & -7 \\ 21 & 28 & 35 \\ -7 & 35 & 14 \end{bmatrix} + \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

$-11+7+21$

$-11+28$

$$A^{-1} = \frac{1}{69}\begin{bmatrix} 17 & 11 & -19 \\ 11 & -1 & 8 \\ -19 & 8 & 5 \end{bmatrix}$$

Diagonalization of a matrix

A matrix "A" is diagonalizable if their exists an invertable matrix "P" (modal matrix) such that $\boxed{D = P^{-1}AP}$ where D is a diagonal matrix. Also the matrix P is said to transform "A" to diagonal form or diagonalize.

Working rule:- To diagonalize the matrix and compute A^n

* To find eigen values of the square matrix "A".
* To find corresponding eigen vectors and write the model matrix "P". i.e, $P = [x_1, x_2, x_3, \ldots x_n]$.
* Calculate $P^{-1} = \frac{\text{adj} P}{|P|}$
* find the diagonal matrix "D" i.e $D = P^{-1}AP$
* Compute $\boxed{A^n = P^{-1}D^nP}$ (or) $\boxed{A^n = PD^nP^{-1}}$

Problems: Working Rule: to diagonalize the matrix A

step1: Determine the Eigen values of a square matrix A.

step2: Determine corresponding Eigen vectors and write the model matrix i.e., $P = [x_1, x_2, \dots x_n]$

step3: calculate P^{-1}.

$$P^{-1} = \frac{\text{adj}.\,P}{|P|}$$

step4: find the diagonal matrix D.

$$D = P^{-1}AP$$

Step 5: Compute A^n from $A^n = P D^n P^{-1}$

problems:

② Diagonalize the matrix $A=\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

Given matrix $A=\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

To find Eigen values

The characteristics eqn is

$$|A-\lambda I|=0$$

$$\left| \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(3-\lambda)-2=0$$

$$12-4\lambda-3\lambda+\lambda^2-2=0$$

$$\lambda^2-7\lambda+10=0$$

$$\lambda^2-2\lambda-5\lambda+10=0$$

$$\lambda(\lambda-2)-5(\lambda-2)=0$$

$$(\lambda-5)(\lambda-2)=0$$

$$\lambda-5=0 \quad \lambda-2=0$$

$$\lambda=5\ , \ \lambda=2$$

$$\therefore \lambda=2,5$$

To find eigen vectors

case(i):- $\lambda = 2$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x + y = 0$$

$$2x + y = 0$$

$$2x = -y$$

$$\frac{x}{1} = \frac{-y}{2} = k$$

$$x = k \;,\; y = -2k$$

$$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

case(ii):- $\lambda = 5$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + y = 0$$

$$2x - 2y = 0$$

$$-x = -y$$

$$\frac{x}{1} = \frac{y}{1} = k$$

$$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

model matrix P is

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{adj.P}{|P|}$$

$$|P| = 1 + 2$$

$$= 3$$

$$adj.P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^T$$

$$adj.P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$D = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

4) Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$. Hence find a) A^8 b) A^4.

A) Given matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$$

To find Eigen values

The characteristics eqn is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(3-\lambda)-4]-1[4]+1[4(2-\lambda)]=0$$

$$(1-\lambda)[6-2\lambda-3\lambda+\lambda^2-4]-4+8-4\lambda=0$$

$$(1-\lambda)[\lambda^2-5\lambda+2]\cancel{-4}+4(1-\lambda)=0$$

$$(1-\lambda)\{[\lambda^2-5\lambda+2]+4\}=0$$

$$(1-\lambda)[\lambda^2-5\lambda+6]=0$$

$$(1-\lambda)[\lambda^2-2\lambda-3\lambda+6]=0$$

$$(1-\lambda)[\lambda(\lambda-2)-3(\lambda-2)]=0$$

$$(1-\lambda)(\lambda-3)(\lambda-2)=0$$

$$\therefore \lambda = 1, 2, 3$$

Case (i): $\lambda=1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y+z=0 \rightarrow (1)$

$y+z=0 \rightarrow (2)$

$-4x+4y+2z=0 \rightarrow (3)$

solving eqn (2) & (3)

	x	y	z	
1	1	0	1	
4	2	-4	4	

$$\frac{x}{2-4} = \frac{y}{-4} = \frac{z}{4} = k$$

$$\frac{x}{-2} = \frac{y}{-4} = \frac{z}{4} = k$$

$$x = k, \quad y = 2k, \quad z = -2k$$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Case(ii):- $\lambda = 2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + y + z = 0 \rightarrow (1)$$

$$z = 0 \rightarrow (2)$$

$$-4x + 4y + z = 0 \rightarrow (3)$$

solving eqn (1) & (3)

$$\begin{array}{cccc} x & y & z & \\ 1 & 1 & -1 & 1 \\ 4 & 1 & -4 & 4 \end{array}$$

$$\frac{x}{1-4} = \frac{y}{-4+1} = \frac{z}{-4+4} = k$$

$$\frac{x}{-3} = \frac{y}{-3} = \frac{z}{0} = k$$

$$x = \cancel{0}k \quad y = k \quad z = 0$$

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

case(iii):- $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x+y+z=0 \rightarrow (1)$$
$$-y+z=0 \rightarrow (2)$$
$$-4x+4y=0 \rightarrow (3)$$

solving eqn (1) & (2)

$$\begin{array}{cccc} x & y & z & \\ 1 & 1 & -2 & 1 \\ -1 & 1 & 0 & -1 \end{array}$$

$$\frac{x}{1+1} = \frac{y}{2} = \frac{z}{2} = k$$

$$x = k, \quad y = k, \quad z = k$$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Modal matrix

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{adj.P}{|P|}$$

$$|P| = 1(1-0) - 1(2+2) + 1(0+2)$$
$$= 1-4+2$$
$$|P| = -1$$

$$adj.P = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 3 & -2 \\ 0 & 1 & -1 \end{bmatrix}^T \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -4 & 3 & 1 \\ 2 & -2 & -1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$D = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

$$D=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$$

To find A^8

$$A^8=PD^8P^{-1}$$

$$D^8=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}^8$$

$$=\begin{bmatrix}1&0&0\\0&256&0\\0&0&6561\end{bmatrix}$$

$$A^8=\begin{bmatrix}1&1&1\\2&1&1\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&256&0\\0&0&6561\end{bmatrix}\begin{bmatrix}-1&1&0\\4&-3&-1\\-2&2&1\end{bmatrix}$$

$$A^8=\begin{bmatrix}-12099&12355&6305\\-12100&12356&6305\\-13120&13120&6561\end{bmatrix}$$

To find A^4

$$D^4=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}^4\Rightarrow\begin{bmatrix}1&0&0\\0&16&0\\0&0&81\end{bmatrix}$$

$$A^4=\begin{bmatrix}1&1&1\\2&1&1\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&16&0\\0&0&81\end{bmatrix}\begin{bmatrix}-1&1&0\\4&-3&-1\\-2&2&1\end{bmatrix}$$

$$A^4=\begin{bmatrix}-99&115&65\\-100&116&65\\-160&-160&81\end{bmatrix}$$

5) Diagonalize the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find A^4 using modal matrix P?

A) Given matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristics eqn is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(5-\lambda)(3-\lambda) - 1] + 1[-(3-\lambda)+1] + 1[1-(5-\lambda)] = 0$$

$$(3-\lambda)[15 - 5\lambda - 3\lambda + \lambda^2 - 1] + [-3+\lambda+1] + [1-5+\lambda] = 0$$

$$(3-\lambda)[\lambda^2 - 8\lambda + 14] + [\lambda - 2] + [\lambda - 4] = 0$$

$$(3-\lambda)[\lambda^2 - 8\lambda + 14] + (2\lambda - 6) = 0$$

$$(3-\lambda)[\lambda^2 - 8\lambda + 14] + 2(\lambda - 3) = 0$$

$$(3-\lambda)[\lambda^2 - 8\lambda + 14] - 2(3-\lambda) = 0$$

$$(3-\lambda)[\lambda^2 - 8\lambda + 14 - 2] = 0$$

$$(3-\lambda)[\lambda^2 - 8\lambda + 12] = 0$$

$$(3-\lambda)[\lambda^2 - 6\lambda - 2\lambda + 12] = 0$$

$$(3-\lambda)[\lambda(\lambda-2) - 6(\lambda-2)] = 0$$

$$(3-\lambda)(\lambda-6)(\lambda-2) = 0$$

$$\therefore \lambda = 2, 3, 6.$$

case(i): $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-y+z=0 \rightarrow (1)$$
$$-x+2y-z=0 \rightarrow (2)$$
$$x-y=0 \rightarrow (3)$$

from eqn (3)

$$x-y=0$$
$$\frac{x}{1}=\frac{y}{1}$$

from eqn (2)

$$-y+2y-z=0$$
$$-z=-y$$
$$\frac{z}{1}=\frac{y}{1}$$

$$X_1=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

case(ii):- $\lambda=2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x-y+z=0 \rightarrow (1)$$
$$-x+4y-z=0 \rightarrow (2)$$
$$x-y+z=0 \rightarrow (3)$$

solving eqn (1) & (2)

$$\begin{matrix} & x & & y & & z & \\ -1 & & 1 & & 1 & & -1 \\ 4 & & -1 & & -1 & & 4 \end{matrix}$$

$$\frac{x}{1-4}=\frac{y}{-1+1}=\frac{z}{4-1}$$

$$\frac{x}{-3}=\frac{y}{0}=\frac{z}{3}=k$$

$$x=k, \ y=0, \ z=-k$$

$$X_2=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=k\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case(iii) : $\lambda = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x - y + z = 0 \rightarrow (1)$

$-x - y - z = 0 \rightarrow (2)$

$x - y - 3z = 0 \rightarrow (3)$

solving eqn (1) & (2)

$$\begin{array}{cccccc} & x & & y & & z \\ -1 & & 1 & & -3 & & -1 \\ -1 & & -1 & & -1 & & -1 \end{array}$$

$$\frac{x}{1+1} = \frac{y}{-1-3} = \frac{z}{3-1} = k$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2} = k$$

$x = k, \; y = -2k, \; z = k$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Modal matrix

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{adj.P}{|P|}$$

$|P| = 1(0-2) - 1(1+2) + 1(-1)$

$= -2 - 3 - 1$

$= -6$

$$adj.P = \begin{bmatrix} -2 & -2 & -2 \\ -3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

expression

$$P^{-1} = -\frac{1}{6}\begin{bmatrix} -2 & -2 & -2 \\ -3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Now, $D = P^{-1}AP$

$$D = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 18 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Now, $A^4 = PD^4P^{-1}$

$$D^4 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}^4$$

$$D^4 = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1296 \end{bmatrix}$$

$$A^4 = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1506 & -2430 & 1410 \\ -2430 & 5346 & -2430 \\ 1410 & -2430 & 1506 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$$

Quadratic forms :

A homogeneous expression of the second degree in any number of variables is called a quadratic form.

An expression of the form $Q = X^T A X$ is called a quadratic forms in 'n' variables $[x_1, x_2 \ldots x_n]$ if the constants are real numbers is called a real quadratic form.

Matrix of quadratic form :

The symetric matrix A is called the matrix of the quadratic form Q and det A is called discriminant of the quadratic form.

i) If $|A| = 0$, then the quadratic form is singular.

ii) If $|A| \neq 0$, then the quadratic form is Non-singular.

Working Rule to the matrix of a quadratic form :

Step-1 : In the first row we write coefficience of x^2, $\frac{1}{2}$ coefficient of xy, $\frac{1}{2}$ coefficient of xz etc., Step-2: In the second row, we write ½ coefficient of yx, coefficient of y^2, ½ coefficient of yz etc..

Step-3: In the third row, we write ½ coefficient of zx, ½ coefficient of zy, coefficient of z^2 etc.

① Find the symmetric matrix corresponding to the quadratic form $x_1^2+6x_1x_2+5x_2^2$.

A) Given quadratic form

$$x_1^2+6x_1x_2+5x_2^2$$

let A be the symmetric matrix of the quadratic form

Then
$$A = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & x_1^2 & x_1x_2/2 \\ x_2 & \frac{x_2x_1}{2} & x_2^2 \end{array}$$

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$$

we can verify, $Q = X^TAX$

let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $X^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$$Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+3x_2 & 3x_1+5x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= (x_1+3x_2)x_1+(3x_1+5x_2)x_2$$

$$= x_1^2+3x_1x_2+3x_1x_2+5x_2^2$$

$$= x_1^2+6x_1x_2+5x_2^2$$

2) find the matrix corresponding to the quadratic form

i) $x^2+y^2+z^2+4xy-2yz+6xz$

ii) $x^2+2y^2-7z^2-4xy+8xz+5yz$

iii) $3x_1^2-2x_2^2-x_3^2-4x_1x_2+12x_2x_3+8x_1x_3$

iv) $x^2+4xy+2y^2$

i) Given quadratic form

$$x^2+y^2+z^2+4xy-2yz+6xz$$

The matrix of corresponding quadratic form is

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & x^2 & xy/2 & xz/2 \\ y & yx/2 & y^2 & yz/2 \\ z & zx/2 & zy/2 & z^2 \end{array}$$

$\therefore$ The required matrix is

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

ii) Given quadratic form

$$x^2+2y^2-7z^2-4xy+8xz+5yz$$

The matrix of corresponding quadratic form is

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & x^2 & xy/2 & xz/2 \\ y & \frac{yx}{2} & y^2 & yz/2 \\ z & \frac{zx}{2} & \frac{zy}{2} & z^2 \end{array}$$

$\therefore$ The required matrix is

$$=\begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 5/2 \\ 4 & 5/2 & -7 \end{bmatrix}$$

iii) Given Quadratic form

$$3x_1^2-2x_2^2-x_3^2-4x_1x_2+12x_2x_3+8x_1x_3$$

The matrix of corresponding quadratic form is

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & x_1^2 & x_1x_2/2 & x_1x_3/2 \\ x_2 & x_2x_1/2 & x_2^2 & x_2x_3/2 \\ x_3 & x_3x_1/2 & x_3x_2/2 & x_3^2 \end{array}$$

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}$$

iv) Given quadratic form is

$$x^2 + 4xy + 2y^2$$

The matrix of corresponding for quadratic form is

	x	y
x	x^2	$xy/2$
y	$yx/2$	y^2

∴ The required matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

3) find the quadratic form corresponding to the matrix

338 – 10, 11, 12, 13, 14

i) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 4 & -1 & 3 \end{bmatrix}$

ii) $\begin{bmatrix} 2 & -3 & 5 \\ -3 & 2 & -2 \\ 5 & -2 & 2 \end{bmatrix}$

iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

i) Given matrix $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 4 & -1 & 3 \end{bmatrix}$

The quadratic form of given matrix is

$$Q = X^T A X$$

let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X^T = [x \; y \; z]$

$$Q = [x \quad y \quad z] \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= [x+4z \quad -2y-z \quad 4x-y+3z]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= (x+4z)x + (-2y-z)y + (4x-y+3z)z$

$= x^2 + 4xz - 2y^2 - zy + 4xz - yz + 3z^2$

$\therefore Q = x^2 - 2y^2 + 3z^2 + 8xz - 2zy$

which is the required Q.f

ii) Given matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 2 & -2 \\ 5 & -2 & 2 \end{bmatrix}$$

The quadratic form of given matrix is

$$Q = X^T A X$$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X^T = [x \quad y \quad z]$

14 $$Q = [x \quad y \quad z]\begin{bmatrix} 2 & -3 & 5 \\ -3 & 2 & -2 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$[2x-3y+5z \quad -3x+2y-2z \quad 5x-2y+2z]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= (2x-3y+5z)x + (-3x+2y-2z)y + (5x-2y+2z)z$

$= 2x^2 - 3xy + 5xz - 3xy + 2y^2 - 2yz + 5xz - 2yz + 2z^2$

$Q = 2x^2 + 2y^2 + 2z^2 - 6xy + 10xz - 4yz$

$\therefore$ which is the required Q.f

iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

The quadratic form of given matrix is

$$Q = X^T A X$$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X^T = [x \quad y \quad z]$

$$Q = [x \quad y \quad z]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x & 3y-z & -y+3z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= x^2+3y^2-zy-yz+3z^2$

$Q. = x^2+3y^2+3z^2-2yz$

$\therefore$ which is required Q.f

4) write the quadratic form corresponding to the symmetric matrix $\begin{bmatrix} 0 & 5/2 & 3 \\ 5/2 & 7 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

A) Given matrix $A = \begin{bmatrix} 0 & 5/2 & 3 \\ 5/2 & 7 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

The quadratic form of given matrix

$$Q = X^T A X$$

$$Q = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 0 & 5/2 & 3 \\ 5/2 & 7 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\frac{5}{2}+\frac{5}{2}$

$\frac{10}{2}$

$$Q = \begin{bmatrix} \frac{5}{2}y+3z & \frac{5}{2}x+7y+z & 3x+y+2z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= (\frac{5}{2}y+3z)x + (\frac{5}{2}x+7y+z)y + (3x+y+2z)z$

$= \frac{5}{2}xy + 3xz + \frac{5}{2}xy + 7y^2 + zy + 3xz + yz + 2z^2$

$= 7y^2+2z^2+5xy+6xz+2zy$

Nature of Quadratic form:

The Quadratic form $Q = X^TAX$, in 'n' variables is said to be

i) positive definite : If all the Eigen values of A are positive.

ii) Negative definite : If all the Eigen values of A are Negative.

iii) positive semi definite: if all the eigen values of A are positive and atleast one Eigen value is zero.

iv) Negative semi definite: if all the Eigen values of A are Negative and atleast one Eigen value is zero.

v) Indefinite: if some eigen values of A are positive and some Eigen values ~~are~~ of A are Negative.

1) find the Nature of the quadratic form $x^2+y^2+z^2-2xy$

A) Given Q.F is $x^2+y^2+z^2-2xy$

The matrix of given Q.F is

	x	y	z
x	x^2	$xy/2$	$xz/2$
y	$yx/2$	y^2	$yz/2$
z	$zx/2$	$zy/2$	z^2

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic eqn of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(1-\lambda)^2]+1(-1(1-\lambda))=0$ $\quad \therefore a^3-b^3\cancel{+}3a^2b+3ab^2$

$(1-\lambda)^3-(1-\lambda)=0$

$1-\lambda^3\cancel{+}3\lambda+3\lambda^2-1+\lambda=0$

$\cancel{-\lambda^3 \; -\lambda^3 \; 3\lambda^2+4\lambda=0}$ $\quad -\lambda^3+3\lambda^2-2\lambda=0$

$\cancel{\lambda^3+3\lambda^2-4\lambda=0}$ $\quad \lambda^3-3\lambda^2+2\lambda=0$

$\cancel{\lambda(\lambda^2+3\lambda-4)=0}$ $\quad \lambda(\lambda^2-3\lambda+2)=0$

$\cancel{\lambda[\lambda+4\lambda-\lambda-4]=0}$ $\quad \lambda[\lambda\cancel{+}2\lambda-\lambda+2]=0$

$\lambda[\lambda(\lambda-2)-1(\lambda-2)]=0$ $(a-b)^2$

$\lambda[(\lambda-2)(\lambda-1)]=0$ a^2+b^2-2ab

$(1-\lambda)(1+\lambda^2-2\lambda-1)$

$\lambda=0,\ \lambda=2,\ \lambda=1$ $(1-\lambda)(\lambda^2-2\lambda)$

$\therefore \lambda=0,1,2$ $\lambda^2-2\lambda-\lambda^3+2\lambda^2$

Here, the ~~nat~~ nature of the given Q. f is positive semi definite. $-\lambda^3+3\lambda^2-2\lambda$

$\lambda^3-3\lambda^2+2\lambda$

$\lambda(\lambda^2-3\lambda+2)$

2) find the nature of QF $2x^2+2y^2+2z^2+2yz$. $\lambda(\lambda^2-2\lambda-\lambda+2)$

Given Q.F is $2x^2+2y^2+2z^2+2yz$ $\lambda((\lambda-2)(\lambda-1)$

The matrix of QF is

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & x^2 & xy/2 & xz/2 \\ y & yx/2 & y^2 & yz/2 \\ z & zx/2 & zy/2 & z^2 \end{array}$$

342 $\downarrow$ Ex: 4, 5

343 $\downarrow$ I $\rightarrow$ 1, 2, 3, 5
II $-$ 5

$$A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The characteristic eqn is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix}=0$$

$(2-\lambda)[(2-\lambda)^2-1]=0$

$(2-\lambda)[4+\lambda^2-4\lambda-1]=0$

$(2-\lambda)[\lambda^2-4\lambda+3]=0$

$(2-\lambda)[\lambda^2-3\lambda-\lambda+3]=0$

$(2-\lambda)[\lambda(\lambda-3)-1(\lambda-3)]=0$

$(2-\lambda)(\lambda-1)(\lambda-3)=0$

$\lambda=2,\ \lambda=1,\ \lambda=3$

$\therefore \lambda=1,2,3$

$\therefore$ Here, the nature of the given Q.f is positive definite.

3) find the nature of Qf $x^2+4xy+6xz-y^2+2yz+4z^2$

Given Q.F is $x^2+4xy+6xz-y^2+2yz+4z^2$

The matrix of Q.F is

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 1\\ 3 & 1 & 4\end{bmatrix}$$

The characteristic eqn is $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & 2 & 3\\ 2 & -1-\lambda & 1\\ 3 & 1 & 4-\lambda\end{vmatrix}=0$$

$(1-\lambda)[(-1-\lambda)(4-\lambda)-1]-2[2(4-\lambda)-3]+3[2-3(-1-\lambda)]=0$

$(1-\lambda)[-4+\lambda-4\lambda+\lambda^2-1]-2[8-2\lambda-3]+3[2+3+3\lambda]=0$

$(1-\lambda)[\lambda^2-3\lambda-5]-2[-2\lambda+5]+3[3\lambda+5]=0$

$\lambda^2-3\lambda-5-\lambda^3+3\lambda^2+5\lambda+4\lambda-10+9\lambda+15=0$

$-\lambda^3+4\lambda^2+15\lambda=0$

$\lambda^3-4\lambda^2-15\lambda=0$

$\lambda(\lambda^2-4\lambda-15)=0$

$\therefore \lambda=0$

Thus, the given quadratic form is indefinite.

4) find the nature of Q.F $2x^2+2y^2+2z^2-2xy-2yz-2zx$.

A) Given Q.F is $2x^2+2y^2+2z^2-2xy-2yz-2zx$

The matrix of QF is

$$A=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$$

The characteristic eqn of A is $|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda & -1 & -1\\ -1 & 2-\lambda & -1\\ -1 & -1 & 2-\lambda\end{vmatrix}=0$$

$$2-\lambda[(2-\lambda)^2-1]+1[-1(2-\lambda)-1]-1[1+(2-\lambda)]=0$$

$$(2-\lambda)[4+\lambda^2-4\lambda-1]-2+\lambda-1-1-2+\lambda=0$$

$$(2-\lambda)[\lambda^2-4\lambda+3]+2\lambda-6=0$$

$$(2-\lambda)[\lambda^2-3\lambda-\lambda+3]+2(\lambda-3)=0$$

$$(2-\lambda)(\lambda-3)(\lambda-1)+2(\lambda-3)=0$$

$$(\lambda-3)[(2-\lambda)(\lambda-1)+2]=0$$

$$(\lambda-3)[2\lambda-2-\lambda^2+\lambda+2]=0$$

$$(\lambda-3)[-\lambda^2+3\lambda]=0$$

$$(\lambda-3)[\lambda(-\lambda+3)]=0$$

$$\therefore \lambda=0,3,3$$

Thus the given Q.F is semi positive definite.

Rank of a Quadratic form

Let $Q = X^T A X$ be a quadratic form in a 'n' variables. The rank of A is called the Rank of Quadratic form.

Normal form (or) canonical form of a quadratic form

let Q be a quadratic form in 'n' variables. Then there exist a Non-singular linear transformation $X = PY$ which transforms $X^T A X$ to Another quadratic form of type $Y^T D Y$ is called a canonical form or Normal of a quadratic form.

Index of a Quadratic form:

The No. of positive terms in a canonical form is called a Index of a ~~positive form~~ quadratic form. It is denoted by 'p'.

Signature of a quadratic form:

The difference b/w no. of positive terms and no. of negative terms in a canonical form is called a signature of Quadratic form. It is denoted by 'S'.

$\boxed{S = 2p - r}$

orthogonal transformation:

If A is an orthogonal matrix and x, y are two column vectors then the transformation $y = Ax$ is called a orthogonal transformation.

problems

(1) Reduce the Quadratic form to canonical form by orthogonal transformation.

If, In the transformation $x = PY$, P is an orthogonal matrix and if $x = PY$ transforms the quadratic form (Q) to the canonical form then Q is said to be reduced to the canonical form by an orthogonal transformation.

Step-1: write the Coefficient matrix A associated with the given quadratic form.

step-2: find the Eigen values of A.

step-3: find the Eigen vectors of A corresponding to the Eigen values such that the eigen vectors must be pairwise orthogonal.

Step-4: find the Normalized Eigen vectors.

$e_1 = \frac{X_1}{|X_1|}$, $e_2 = \frac{X_2}{|X_2|}$ & $e_3 = \frac{X_3}{|X_3|}$

Step 5: write the orthogonal matrix $P = [e_1, e_2\ e_3 \cdots e_n]$

Step-6: write the Diagonalized matrix $D = P^{-1}AP$ $\because [P^{-1} = P^T]$

then $\boxed{D = P^TAP}$

Step-7: write the canonical form $[Y^TDY]$

2) Reduce the Q.F $3x^2+5y^2+3z^2-2xy-2yz+2zx$ to canonical form by orthogonal reduction and hence state nature, ~~a~~ Rank, Index and signature.

The matrix of the Q.F is

$$= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic eqn is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 3+5+3 = 11$

$S_2 = 14 + 8 + 14 = 36$

$S_3 = 36$

$\cancel{\lambda^3 - 11\lambda^2 + 42\lambda - 36 = 0}$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

2	1	-11	36	-36
	0	2	-18	18
	1	-9	18	$\boxed{0}$

(6, 2, 3)

$(\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$

$(\lambda-2)(\lambda^2 - 3\lambda - 6\lambda + 18) = 0$

$(\lambda-2)(\lambda(\lambda-3) - 6(\lambda-3)) = 0$

$(\lambda-2)(\lambda-6)(\lambda-3) = 0$

$\therefore \lambda = 2, 3, 6$

To find Eigen vectors

case (i): $\lambda = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~x y z~~

$x - y + z = 0 \rightarrow (1)$

$-x + 3y - z = 0 \rightarrow (2)$

$x - y + z = 0 \rightarrow (3)$

Solving eqn (1) & (2)

$(1)\frac{4 \times 3}{42}$ 15-1

15-1

$(1)\frac{14}{42}$

$3(15-1) + 1(-3+1) + 1(1-5)$

$3(14) + 1(-2) + 1(-4)$

$42 - 2 - 4$

$42 - 6$

36

376 ↓ Ex: 4

381 ↓ Ex 16

388 ↓ Ex: 8

$$\begin{array}{cccccc} & x & & y & & z \\ -1 & & 1 & & 1 & & -1 \\ 3 & & -1 & & -1 & & 3 \end{array}$$

$$\frac{x}{1-3} = \frac{y}{-1+1} = \frac{z}{3-1}$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2} = k$$

$$x = -k, \quad y = 0, \quad z = k$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

<u>case(ii)</u>:- $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-y + z = 0 \quad \rightarrow (1)$$

$$-x + 2y - z = 0 \quad \rightarrow (2)$$

$$x - y = 0 \quad \rightarrow (3)$$

solving eqn (1) & (2)

$$\begin{array}{cccccc} & x & & y & & z \\ -1 & & 1 & & 0 & & -1 \\ 2 & & -1 & & -1 & & 2 \end{array}$$

$$\frac{x}{1-2} = \frac{y}{-1} = \frac{z}{0-1}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1} = k$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

<u>case(iii)</u>:- $\lambda = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x - y + z = 0 \rightarrow (1)$

$-x - y - z = 0 \rightarrow (2)$

$x - y - 3z = 0 \rightarrow (3)$

solving eqn (1) & (2)

$$\begin{array}{cccc} x & y & z & \\ -1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -1 \end{array}$$

$$\frac{x}{1+1} = \frac{y}{-1-3} = \frac{z}{3-1}$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2} = k$$

$x = k, \; y = -2k, \; z = \cancel{\phi} k$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

let $\|X_1\|, \|X_2\|$ & $\|X_3\|$ are normalized vectors

$\|X_1\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

$\|X_2\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\|X_3\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

Normalized eigen vectors

$$e_1 = \frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$e_2 = \frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$e_3 = \frac{X_3}{\|X_3\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

orthogonal matrix

$P = [e_1 \, e_2 \, e_3]$

spectral matrix $D = P^{-1}AP$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

spectral matrix $D = P^{-1}AP$

w.k.T $P^{-1} = P^T$

$D = P^T A P$

$$P^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$\therefore D = P^T A P$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -2/\sqrt{2} & 0 & 2/\sqrt{2} \\ 3/\sqrt{3} & 3/\sqrt{3} & 3/\sqrt{3} \\ 6/\sqrt{6} & -12/\sqrt{6} & 6/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

canonical form $y^T D y$

let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ & $y^T = [y_1 \ y_2 \ y_3]$

$$[y_1 \ y_2 \ y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

rank $r = 3$

Index $P = 3$

signature $s = 2P - r$

$= 2(3) - 3$

$= 6 - 3$

$s = 3$

$$\begin{bmatrix} 2y_1 & 3y_2 & 6y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$2y_1^2 + 3y_2^2 + 6y_3^2$

∴ which is the required canonical form.

3) Reduce the Q.F $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ into sum of squares form by an orthogonal transformation and given the matrix of transformation.

The matrix of given Q.F is

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic eqn is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 9$

$S_2 = 8 + 8 + 8 = 24$

$S_3 = 16$

$3(8) - 1(3+1) + 1(-1-$

$24 - 4 - 4$

$24 - 8$

16

$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

By synthetic Division

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline & 1 & -8 & 16 & \boxed{0} \end{array}$$

$(\lambda - 1)(\lambda^2 - 8\lambda + 16) = 0$

$(\lambda - 1)(\lambda^2 - 4\lambda - 4\lambda + 16) = 0$

$(\lambda - 1)(\lambda(\lambda - 4) - 4(\lambda - 4)) = 0$

$(\lambda - 1)(\lambda - 4)(\lambda - 4) = 0$

$\therefore \lambda = 1, 4, 4$

To find Eigen vectors

case (i) :- $\lambda = 1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 - R_1$

$R_2 \to 2R_2 - R_1$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x + y + z = 0$

$3y - 3z = 0$

Let $\boxed{z = k}$

$3y - 3k = 0$

$3y = 3k$

$\boxed{y = k}$

$2x + k + k = 0$

$2x = -2k$

$\boxed{x = -k}$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

case (ii) :- $\lambda = 4$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + R_1$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + y + z = 0$

let $y = k_2$ & $z = k_1$

$-x + k_2 + k_1 = 0$

$x = k_1 + k_2$

$$X = \begin{bmatrix} k_1 + k_2 \\ k_2 \\ k_1 \end{bmatrix} \Rightarrow k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

let $\|x_1\|$, $\|x_2\|$ & $\|x_3\|$ are normalized vectors

$\|x_1\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$

$\|x_2\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$\|x_3\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

Normalized Eigen vectors

$$e_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$e_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$e_3 = \frac{x_3}{\|x_3\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

orthogonal matrix

$P = [e_1 \ e_2 \ e_3]$

Quadratic form

$$P = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

spectral matrix $D = P^{-1}AP$

w.k.T $P^{-1} = P^T$

$$P^T = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$D = P^T A P$

$$D = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$